Stochastic maximum principle for stochastic recursive optimal control problem under volatility ambiguity

Mingshang Hu [*]   Shaolin Ji[†]

September 1, 2015

Abstract. We study a stochastic recursive optimal control problem in which the cost functional is described by the solution of a backward stochastic differential equation driven by G-Brownian motion. Some of the economic and financial optimization problems with volatility ambiguity can be formulated as such problems. Different from the classical variational approach, we establish the maximum principle by the linearization and weak convergence methods.

Key words. Backward stochastic differential equations, Volatility ambiguity, G-expectation, Maximum principle, Robust control

AMS subject classifications. 93E20, 60H10, 35K15

1 Introduction

In economic theory, for a given consumption process $(c_s)_{0 \leq s \leq T}$ under probability P, Duffie and Epstein [5] introduced the stochastic differential recursive utility

$$y^P(t) = E_P[\int_t^T f(y^P(s), c(s))ds \mid \mathcal{F}_t], \ 0 \leq t \leq T, \tag{1.1}$$

and many optimization problems for the stochastic differential recursive utilities are well studied by Duffie and Skiadas [6] etc. In fact, the stochastic differential recursive utility is associated with the solution of a particular backward stochastic differential equation (BSDE). It is well known that the general BSDE was introduced by Pardoux and Peng [22]. Peng [24] first generalized the classical stochastic optimal control problem to a new one in which the objective functional is defined by the solution of the following BSDE

1

(1.2) at time 0:

$$\begin{cases} -dy(t) = & f(t, x(t), y(t), z(t), u(t))dt - z(t)dB(t), \\ y(T) = & \phi(x(T)), \end{cases} \tag{1.2}$$

where B is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. From the BSDE point of view, El Karoui et. al. [8] considered a more general class of recursive utilities defined as the solution of BSDEs. Thus, this new kind of stochastic optimal control problem is called the stochastic recursive optimal control problem.

Chen and Epstein [3] studied the stochastic differential recursive utility with drift ambiguity. The drift ambiguity in their context is described by a class of equivalent probability measures $\mathcal{P}$. The stochastic differential recursive utility with drift ambiguity is defined as the lower envelope

$$y(t) = \operatorname*{ess\,inf}_{P \in \mathcal{P}} y^P(t), \ 0 \le t \le T, \tag{1.3}$$

where $y^P(t)$ is the solution of (1.1) at time t. They proved that $y(t)$ of (1.3) can be characterized by a special BSDE and the corresponding recursive utility optimization problems with drift ambiguity still fall in the framework of the stochastic recursive optimal control problem.

Many economic and financial problems involve volatility ambiguity (for the motivation to consider volatility uncertainty, refer to Epstein and Ji [10, 11]). It is well known that volatility ambiguity is chracterized by a family of nondominated probability measures. In this case, (1.3) can not be formulated as a classical BSDE, because it can not be modeled within a probability space framework. So we need a new framework to accommodate stochastic differential recursive utility with volatility ambiguity.

Inspired by studying financial problems with volatility ambiguity (see [1, 20]), Peng introduced a fully nonlinear expectation, called G-expectation $\hat{\mathbb{E}}[\cdot]$ (see [28] and the references therein) which does not require a probability space framework. Under this G-expectation framework (G-framework for short) a new type of Brownian motion called G-Brownian motion was constructed. The stochastic calculus with respect to the G-Brownian motion has been established.

Recently, Hu et. al developed the BSDE theory under this G-expectation framework in [15, 16] (see Soner et al. [30] for another formulation of fully nonlinear BSDE, called 2BSDE). In more details, they proved that the following BSDE driven by G-Brownian motion (G-BSDE for short)

$$\begin{aligned} y(t) = & \ \xi + \int_t^T f(s, y(s), z(s))ds + \int_t^T g(s, y(s), z(s))d\langle B\rangle(s) \\ & - \int_t^T z(s)dB(s) - (K(T) - K(t)) \end{aligned}$$

has a unique triple of solution (y, z, K). In fact, in the volatility ambiguity case, (1.3) can be formulated as a special G-BSDE (see [10, 11]). So the stochastic recursive utility optimization problem with volatility ambiguity is a special case of the following problem (1.4). The state equations are the following forward and backward SDEs driven by G-Brownian motion: for $t \in [0, T]$,

$$\begin{cases} dx(t) = & b(t, x(t), u(t))dt + h^{ij}(t, x(t), u(t))d\langle B^i, B^j\rangle(t) + \sigma^i(t, x(t), u(t))dB^i(t), \\ x(0) = & x_0 \in \mathbb{R}^n, \end{cases}$$

$$
\begin{cases}
-dy(t) = & f(t, x(t), y(t), z(t), u(t))dt + g^{ij}(t, x(t), y(t), z(t), u(t))d\langle B^i, B^j\rangle(t) - z(t)dB(t) - dK(t), \\
y(T) = & \phi(x(T)).
\end{cases}
$$
(1.4)

The cost functional is introduced by the solution of the above BSDE at time 0, i.e., $J(u(\cdot)) = y(0)$. The stochastic recursive optimal control problem is to minimize the cost functional over the admissible controls.

The stochastic maximum principle is an important approach to solve stochastic optimal control problems (see [12, 13, 18, 19, 21, 23, 29, 31, 32, 34, 36]). A local form of the stochastic maximum principle for the classical stochastic recursive optimal control problem was first established in Peng [24]. In this paper, we study the stochastic maximum principle for the problem (1.4) when the control domain is convex.

Note that the solution y of (1.4) at time 0 can be written as

$$
y_0 = \hat{\mathbb{E}}[\phi(x(T)) + \int_0^T f(t, x(t), y(t), z(t), u(t))dt + \int_0^T g^{ij}(t, x(t), y(t), z(t), u(t))d\langle B^i, B^j\rangle(t)]
$$

$$
= \sup_{P \in \mathcal{P}} E_P[\phi(x(T)) + \int_0^T f(t, x(t), y(t), z(t), u(t))dt + \int_0^T g^{ij}(t, x(t), y(t), z(t), u(t))d\langle B^i, B^j\rangle(t)], \quad (1.5)
$$

where $\mathcal{P}$ is a family of weakly compact nondominated probability measures (see [4]). Thus, our stochastic recursive optimal control problem is essentially a "inf sup problem". Such problem is known as the robust optimal control problem, i.e., we consider the worst scenario by maximizing over a set of probability measures and then we minimize the cost functional.

For the case f does not depend on (y, z) and $g^{ij} = 0$, i.e.,

$$
J(u(\cdot)) = \hat{\mathbb{E}}[\phi(x(T)) + \int_0^T f(t, x(t), u(t))dt],
$$
(1.6)

Xu [33] studied this problem. Based on the subadditivity of $\hat{\mathbb{E}}[\cdot]$, he obtained the variational inequality by the classical variational method. But he did not get the stochastic maximum principle since the sublinear operator $\hat{\mathbb{E}}$ in his main theorem can not be deleted. It is worth to pointing out that the classical variational method can not be applied to obtain the variational inequality for our problem (1.4).

In the literatures, in order to derive the maximum principle for the classical stochastic recursive optimal control problem, one need to obtain the variational equation for the BSDE (1.2). But in our context, since the K term of the solution of (1.4) is a decreasing G-martingale, it is unable to obtain the "derivative" for K in general. So we can not obtain the variational equation for the G-BSDE (1.4). To overcome this difficulty, we introduce the linearization and weak convergence methods to directly obtain the derivative for the value function. By Minimax Theorem, the variational inequality on a reference probability P^* is obtained. Based on the obtained variational inequality, we derive the stochastic maximum principle holds P^*-a.s.. Furthermore, we prove that the obtained stochastic maximum principle is also a sufficient condition under some convex assumptions.

The paper is organized as follows. In Section 2, we present some fundamental results on G-expectation theory. We formulate our stochastic recursive optimal control problem in Section 3. We derive the maximum principle in Section 4 and give the general results in Section 5. In Section 6, applying the obtained maximum principle, we solve a LQ problem.

2 Preliminaries

We review some basic notions and results of G-expectations. The readers may refer to [16, 25–28] for more details.

Let $\Omega = C_0([0, \infty); \mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued continuous functions on $[0, \infty)$ with $\omega_0 = 0$ and let $(B(t))_{t \geq 0}$ be the canonical process. For each fixed $T > 0$, set

$$L_{ip}(\Omega_T) := \{\varphi(B(t_1), \cdots, B(t_n)) : n \geq 1, t_1, \cdots, t_n \in [0, T], \varphi \in C_{b.Lip}(\mathbb{R}^{d \times n})\},$$

where $C_{b.Lip}(\mathbb{R}^{d \times n})$ denotes the space of bounded Lipschitz functions on $\mathbb{R}^{d \times n}$. Obviously, $L_{ip}(\Omega_T) \subset L_{ip}(\Omega_{T'})$ for $T < T'$. We also set

$$L_{ip}(\Omega) = \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n).$$

For each given monotonic and sublinear function $G(\cdot) : \mathbb{S}_d \to \mathbb{R}$, where $\mathbb{S}_d$ denotes the collection of $d \times d$ symmetric matrices, there exists a bounded and closed subset $\Gamma \subset \mathbb{R}^{d \times d}$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \operatorname{tr}[\gamma \gamma^T A], \tag{2.1}$$

where $\mathbb{R}^{d \times d}$ denotes the collection of $d \times d$ matrices. In this paper we only consider non-degenerate G, i.e., there exists some $\underline{\sigma}^2 > 0$ such that $G(A) - G(B) \geq \underline{\sigma}^2 \operatorname{tr}[A - B]$ for any $A \geq B$. Now, we define a functional $\hat{\mathbb{E}} : L_{ip}(\Omega) \to \mathbb{R}$ by two steps.

Step 1. For $X = \varphi(B(t + s) - B(s))$ with $t, s \geq 0$ and $\varphi \in C_{b.Lip}(\mathbb{R}^d)$, we define

$$\hat{\mathbb{E}}[X] = u(t, 0),$$

where u is the solution of the following G-heat equation:

$$\partial_t u - G(D_{xx}^2 u) = 0, \ u(0, x) = \varphi(x).$$

Step 2. For $X = \varphi(B(t_1) - B(t_0), B(t_2) - B(t_1), \cdots, B(t_n) - B(t_{n-1}))$ with $0 \leq t_0 < \cdots < t_n$ and $\varphi \in C_{b.Lip}(\mathbb{R}^{d \times n})$, we define

$$\hat{\mathbb{E}}[X] = \varphi_n,$$

where φ_n is obtained via the following procedure:

$$\varphi_1(x_1, \cdots, x_{n-1}) = \hat{\mathbb{E}}[\varphi(x_1, \cdots, x_{n-1}, B(t_n) - B(t_{n-1}))],$$

$$\varphi_2(x_1, \cdots, x_{n-2}) = \hat{\mathbb{E}}[\varphi_1(x_1, \cdots, x_{n-2}, B(t_{n-1}) - B(t_{n-2}))],$$

$$\vdots$$

$$\varphi_n = \hat{\mathbb{E}}[\varphi_{n-1}(B(t_1) - B(t_0))].$$

The corresponding conditional expectation $\hat{\mathbb{E}}_t$ of X with $t = t_i$ is defined by

$$\hat{\mathbb{E}}_{t_i}[\varphi(B(t_1) - B(t_0), B(t_2) - B(t_1), \cdots, B(t_n) - B(t_{n-1}))]$$

$$= \varphi_{n-i}(B(t_1) - B(t_0), \cdots, B(t_i) - B(t_{i-1})).$$

It is easy to check that $(\hat{\mathbb{E}}_t)_{t \geq 0}$ satisfies the following properties: for each $X, Y \in L_{ip}(\Omega)$,

4

(i) Monotonicity: If $X \geq Y$, then $\hat{\mathbb{E}}_t[X] \geq \hat{\mathbb{E}}_t[Y]$;

(ii) Constant preservation: $\hat{\mathbb{E}}_t[X] = X$ for $X \in L_{ip}(\Omega_t)$;

(iii) Sub-additivity: $\hat{\mathbb{E}}_t[X + Y] \leq \hat{\mathbb{E}}_t[X] + \hat{\mathbb{E}}_t[Y]$;

(iv) Positive homogeneity: $\hat{\mathbb{E}}_t[XY] = X^+\hat{\mathbb{E}}_t[Y] + X^-\hat{\mathbb{E}}_t[-Y]$ for $X \in L_{ip}(\Omega_t)$;

(v) Consistency: $\hat{\mathbb{E}}_s[\hat{\mathbb{E}}_t[X]] = \hat{\mathbb{E}}_{s \wedge t}[X]$, specially, $\hat{\mathbb{E}}[\hat{\mathbb{E}}_t[X]] = \hat{\mathbb{E}}[X]$.

We denote by $L_G^p(\Omega)$ the completion of $L_{ip}(\Omega)$ under the norm $\|X\|_{p,G} = (\hat{\mathbb{E}}[|X|^p])^{1/p}$ for $p \geq 1$, similarly for $L_G^p(\Omega_T)$. For each $t \geq 0$, $\hat{\mathbb{E}}_t[\cdot]$ can be extended continuously to $L_G^1(\Omega)$ under the norm $\|\cdot\|_{1,G}$. $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$ is called a G-expectation space. The corresponding canonical process $(B(t))_{t\geq 0}$ is called a G-Brownian motion.

Definition 2.1 *A process $(X(t))_{t\geq 0}$ is called a G-martingale if $X(t) \in L_G^1(\Omega_t)$ and $\hat{\mathbb{E}}_s[X(t)] = X(s)$ for $s \leq t$.*

Remark 2.2 *It is important to note that $(-X(t))_{t\geq 0}$ may be not a G-martingale.*

Set
$$\mathcal{P} = \{P : P \text{ is a probability on } (\Omega, \mathcal{B}(\Omega)), \ E_P[X] \leq \hat{\mathbb{E}}[X] \text{ for } X \in L_G^1(\Omega)\}. \tag{2.2}$$

Theorem 2.3 *([4, 17]) Let $\mathcal{P}$ be defined as in (2.2). Then $\mathcal{P}$ is convex, weakly compact and*
$$\hat{\mathbb{E}}[\xi] = \max_{P \in \mathcal{P}} E_P[\xi] \quad \text{for all } \xi \in L_G^1(\Omega).$$

$\mathcal{P}$ is called a set that represents $\hat{\mathbb{E}}$.

The following proposition is important in our paper.

Proposition 2.4 *([4]) Let $\{P_n : n \geq 1\} \subset \mathcal{P}$ converge weakly to P. Then for each $\xi \in L_G^1(\Omega)$, we have $E_{P_n}[\xi] \to E_P[\xi]$.*

Definition 2.5 *Let $M_G^0(0, T)$ be the collection of processes in the following form: for a given partition $\{t_0, \cdots, t_N\} = \pi_T$ of $[0, T]$,*
$$\eta(t) = \sum_{j=0}^{N-1} \xi_j I_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \cdots, N - 1$.

We denote by $M_G^p(0, T)$ the completion of $M_G^0(0, T)$ under the norm $\|\eta\|_{M_G^p} = \{\hat{\mathbb{E}}[\int_0^T |\eta(s)|^p ds]\}^{1/p}$ for $p \geq 1$. The Itô's integral $\int_0^T \eta(s)dB(s)$ is well defined for $\eta \in M_G^2(0, T)$.

3 Stochastic optimal control problem

We first give the definition of admissible controls.

Definition 3.1 $u(\cdot)$ *is said to be an admissible control on* $[0, T]$, *if it satisfies the following conditions:*

(i) $u(\cdot) : [0, T] \times \Omega \to U$ *where* U *is a nonempty convex subset of* $\mathbb{R}^m$;

(ii) $u(\cdot) \in M_G^\beta(0, T; \mathbb{R}^m)$ *with* $\beta > 2$.

The set of admissible controls is denoted by $\mathcal{U}[0, T]$.

In the rest of this paper, we use the Einstein summation convention.

Let $u(\cdot) \in \mathcal{U}[0, T]$. Consider the following forward and backward SDEs driven by G-Brownian motion: for $t \in [0, T]$,

$$\begin{cases} dx(t) = & b(t, x(t), u(t))dt + h^{ij}(t, x(t), u(t))d\langle B^i, B^j \rangle(t) + \sigma^i(t, x(t), u(t))dB^i(t), \\ x(0) = & x_0 \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

$$\begin{cases} -dy(t) = & f(t, x(t), y(t), z(t), u(t))dt + g^{ij}(t, x(t), y(t), z(t), u(t))d\langle B^i, B^j \rangle(t) - z(t)dB(t) - dK(t), \\ y(T) = & \phi(x(T)), \end{cases}$$
$$\tag{3.2}$$

where

$$b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n;$$

$$h^{ij} : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n;$$

$$\sigma = [\sigma^1, \dots, \sigma^d] : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times d};$$

$$f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1 \times d} \times U \to \mathbb{R};$$

$$g^{ij} : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1 \times d} \times U \to \mathbb{R};$$

$$\phi : \mathbb{R}^n \to \mathbb{R}.$$

Denote

$$S_G^0(0, T) = \{h(t, B_{t_1 \wedge t}, \cdots, B_{t_n \wedge t}) : t_1, \dots, t_n \in [0, T], h \in C_{b, Lip}(\mathbb{R}^{n+1})\};$$

$$S_G^2(0, T) = \{\text{the completion of } S_G^0(0, T) \text{ under the norm } \|\eta\|_{S_G^2} = \{\hat{\mathbb{E}}[\sup_{t \in [0, T]} |\eta_t|^2]\}^{\frac{1}{2}}\}.$$

For given $u(\cdot) \in \mathcal{U}[0, T]$, $x(\cdot)$ and $(y(\cdot), z(\cdot), K(\cdot))$ are called solutions of the above forward and backward SDEs respectively if $x(\cdot) \in M_G^2(0, T; \mathbb{R}^n)$; $(y(\cdot), z(\cdot)) \in S_G^2(0, T) \times M_G^2(0, T; \mathbb{R}^{1 \times d})$; $K(\cdot)$ is a decreasing G-martingale with $K(0) = 0$ and $K(T) \in L_G^2(\Omega_T)$; (3.1) and (3.2) are satisfied respectively.

We assume:

(H1) $b, h^{ij}, \sigma, f, g^{ij}, \phi$ are continuous and differentiable in (x, y, z, u);

(H2) The derivatives of $b, h^{ij}, \sigma, f, g^{ij}, \phi$ in (x, y, z, u) are bounded;

(H3) There exists a modulus of continuity $\bar{\omega} : [0, \infty) \to [0, \infty)$ such that for any $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, y, $y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^{1 \times d}$, $u, u' \in \mathbb{R}^m$,

$$|\varphi(t, x, y, z, u) - \varphi(t, x', y', z', u')| \leq \bar{\omega}(|x - x'| + |y - y'| + |z - z'| + |u - u'|),$$

where φ is the derivatives of $b, h^{ij}, \sigma, f, g^{ij}, \phi$ in (x, y, z, u).

We have the following theorems.

Theorem 3.2 *([28]) Let assumptions (H1)-(H2) hold. Then (3.1) has a unique solution $x(\cdot)$.*

Theorem 3.3 *([15]) Let assumptions (H1)-(H2) hold. Then (3.2) has a unique solution $(y(\cdot), z(\cdot), K(\cdot))$.*

The state equation of our stochastic optimal control problem is governed by the above forward and backward SDEs (3.1) and (3.2). The cost functional is introduced by the solution of the BSDE (3.2) at time 0, i.e.,

$$J(u(\cdot)) = y(0).$$

The stochastic optimal control problem is to minimize the cost functional over $\mathcal{U}[0, T]$.

Remark 3.4 *We point out that $\mathcal{U}[0, T]$ contains all feedback controls (see Hu and Ji [14]). In the last section, we show that the optimal control of the LQ problem is a special kind of feedback control.*

In summary, our stochastic control problem is

$$\begin{cases} \text{Minimize} & J(u(\cdot)) \\ \text{subject to} & u(\cdot) \in \mathcal{U}[0, T]. \end{cases}$$

4 Stochastic Maximum Principle

In this section, to ease the presentation we only study the case where $h_{ij} \equiv 0$, $g_{ij} \equiv 0$ and f does not include z term. We will present the results for the general case in Section 5.

4.1 Variational equation

Let $\bar{u}(\cdot)$ be optimal and $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot), \bar{K}(\cdot))$ be the corresponding state processes of (3.1) and (3.2). Take an arbitrary $u(\cdot) \in \mathcal{U}[0, T]$. Since $\mathcal{U}[0, T]$ is convex, then, for each $0 \leq \rho \leq 1$, $\bar{u}(\cdot) + \rho(u(\cdot) - \bar{u}(\cdot)) \in \mathcal{U}[0, T]$. Let $(x_\rho(\cdot), y_\rho(\cdot), z_\rho(\cdot), K_\rho(\cdot))$ be the state processes of (3.1) and (3.2) associated with $\bar{u}(\cdot) + \rho(u(\cdot) - \bar{u}(\cdot))$.

To derive the first-order necessary condition in terms of small ρ, let $\hat{x}(\cdot)$ be the solution of the following SDE:

$$\begin{cases} d\hat{x}(t) = & [b_x(t)\hat{x}(t) + b_u(t)(u(t) - \bar{u}(t))]dt + [\sigma_x^i(t)\hat{x}(t) + \sigma_u^i(t)(u(t) - \bar{u}(t))]dB^i(t), \\ \hat{x}(0) = & 0, \end{cases} \tag{4.1}$$

where $b_x(t) = b_x(t, \bar{x}(t), \bar{u}(t))$, $b_u(t) = b_u(t, \bar{x}(t), \bar{u}(t))$, $\sigma_x^i(t) = \sigma_x^i(t, \bar{x}(t), \bar{u}(t))$, $\sigma_u^i(t) = \sigma_u^i(t, \bar{x}(t), \bar{u}(t))$.

In this paper, we define

$$
b_x(t) = \begin{bmatrix} b_{1x_1}(t), & \cdots & , b_{1x_n}(t) \\ \vdots & & \vdots \\ b_{nx_1}(t), & \cdots & , b_{nx_n}(t) \end{bmatrix}.
$$

The other derivatives are defined similarly.

Equation (4.1) is called the variational equation for SDE (3.1). By Theorem 1.2 in [28], there exists a unique solution $\hat{x}(\cdot) \in M_G^2(0, T; \mathbb{R}^n)$ to equation (4.1).

Set

$$
\tilde{x}_\rho(t) = \rho^{-1}[x_\rho(t) - \bar{x}(t)] - \hat{x}(t).
$$

Proposition 4.1 *Assume (H1)-(H3) hold. Then*

(i) *there exists a positive constant C such that $\hat{\mathbb{E}}[| \tilde{x}_\rho(t) |^2] \leq C$ for $0 \leq \rho \leq 1$;*

(ii) $\lim_{\rho \to 0} \sup_{0 \leq t \leq T} \hat{\mathbb{E}}[| \tilde{x}_\rho(t) |^2] = 0.$

In the following, we always use the constant C for simplicity, where C can be change from line to line. For prove this proposition, we need the following lemma.

Lemma 4.2 *Suppose that η belongs to $M_G^1(0, T)$. Then for each $\varepsilon > 0$, there exists a positive number δ such that $\hat{\mathbb{E}}[\int_0^T | \eta | I_A dt] < \varepsilon$ for any $A \in \mathcal{B}([0, T]) \times \mathcal{F}_T$ with $\hat{\mathbb{E}}[\int_0^T I_A(t, \omega)dt] < \delta$.*

Proof. Since $\eta \in M_G^1(0, T)$, we have

$$
\lim_{N \to \infty} \hat{\mathbb{E}}[\int_0^T | \eta | I_{\{|\eta| \geq N\}} dt] = 0
$$

by Proposition 18 in [4]. Then for any $\varepsilon > 0$, there exists a N_0 such that $\hat{\mathbb{E}}[\int_0^T | \eta | I_{\{|\eta| \geq N_0\}} dt] < \frac{\varepsilon}{2}$. Take $\delta = \frac{\varepsilon}{2N_0}$. For any $A \in \mathcal{B}([0, T]) \times \mathcal{F}_T$ with $\hat{\mathbb{E}}[\int_0^T I_A(t, \omega)dt] < \delta$, we have that

$$
\begin{aligned}
\hat{\mathbb{E}}[\int_0^T | \eta | I_A dt] &= \hat{\mathbb{E}}[\int_0^T | \eta | (I_{\{|\eta| \geq N_0\} \cap A} + I_{\{|\eta| < N_0\} \cap A})dt] \\
&\leq \hat{\mathbb{E}}[\int_0^T | \eta | I_{\{|\eta| \geq N_0\} \cap A} dt] + \hat{\mathbb{E}}[\int_0^T | \eta | I_{\{|\eta| < N_0\} \cap A} dt] \\
&\leq \hat{\mathbb{E}}[\int_0^T | \eta | I_{\{|\eta| \geq N_0\}} dt] + \hat{\mathbb{E}}[\int_0^T N_0 I_{\{|\eta| < N_0\} \cap A} dt] \\
&\leq \varepsilon.
\end{aligned}
$$

This completes the proof. ∎

Proof of Proposition 4.1. (i) From (3.1) and (4.1), we have

$$
\begin{cases}
d\tilde{x}_\rho(t) = & \rho^{-1}[b_\rho(t) - b(t) - \rho(b_x(t)\hat{x}(t) + b_u(t)(u(t) - \bar{u}(t)))]dt \\
& + \rho^{-1}[\sigma_\rho^i(t) - \sigma^i(t) - \rho(\sigma_x^i(t)\hat{x}(t) + \sigma_u^i(t)(u(t) - \bar{u}(t)))]dB^i(t), \\
\tilde{x}_\rho(0) = & 0.
\end{cases}
$$

where $b_\rho(t) = b(t, x_\rho(t), \bar{u}(t)+\rho(u(t)-\bar{u}(t)))$, $b(t) = b(t, \bar{x}(t), \bar{u}(t))$, $\sigma_\rho^i(t) = \sigma^i(t, x_\rho(t), \bar{u}(t)+\rho(u(t)-\bar{u}(t)))$ and $\sigma^i(t) = \sigma^i(t, \bar{x}(t), \bar{u}(t))$. Let

$$A_\rho(t) = \int_0^1 b_x(t, \bar{x}(t) + \lambda\rho(\hat{x}(t) + \tilde{x}_\rho(t)), \bar{u}(t) + \lambda\rho(u(t) - \bar{u}(t)))d\lambda,$$

$$B_\rho^i(t) = \int_0^1 \sigma_x^i(t, \bar{x}(t) + \lambda\rho(\hat{x}(t) + \tilde{x}_\rho(t)), \bar{u}(t) + \lambda\rho(u(t) - \bar{u}(t)))d\lambda,$$

$$C_\rho(t) = [A_\rho(t) - b_x(t)]\hat{x}(t) + \int_0^1 [b_u(t, \bar{x}(t) + \lambda\rho(\hat{x}(t) + \tilde{x}_\rho(t)), \bar{u}(t) + \lambda\rho(u(t) - \bar{u}(t))) - b_u(t)](u(t) - \bar{u}(t))d\lambda,$$

$$D_\rho^i(t) = [B_\rho(t) - \sigma_x(t)]\hat{x}(t) + \int_0^1 [\sigma_u^i(t, \bar{x}(t) + \lambda\rho(\hat{x}(t) + \tilde{x}_\rho(t)), \bar{u}(t) + \lambda\rho(u(t) - \bar{u}(t))) - \sigma_u^i(t)](u(t) - \bar{u}(t))d\lambda.$$

Thus,

$$\begin{cases} d\tilde{x}_\rho(t) = & [A_\rho(t)\tilde{x}_\rho(t) + C_\rho(t)]dt + [B_\rho^i(t)\tilde{x}_\rho(t) + D_\rho^i(t)]dB^i(t), \\ \\ \tilde{x}_\rho(0) = & 0. \end{cases}$$

Using Itô's formula to $| \tilde{x}_\rho(t) |^2$, we get

$$\hat{\mathbb{E}}[| \tilde{x}_\rho(t) |^2]$$
$$= \hat{\mathbb{E}}[\int_0^t 2\langle \tilde{x}_\rho(s), A_\rho(s)\tilde{x}_\rho(s) + C_\rho(s)\rangle ds + \int_0^t \langle B_\rho^i(s)\tilde{x}_\rho(s) + D_\rho^i(s), B_\rho^j(s)\tilde{x}_\rho(s) + D_\rho^j(s)\rangle d\langle B^i, B^j\rangle(s)]$$
$$\leq C(\hat{\mathbb{E}}[\int_0^t | \tilde{x}_\rho(s) |^2 \, ds] + I_\rho),$$

where C is a constant and

$$I_\rho = \hat{\mathbb{E}}[\int_0^T (| C_\rho(s) |^2 + | D_\rho^i(s) |^2)ds].$$

Applying Gronwall's inequality, we obtain that

$$\hat{\mathbb{E}}[| \tilde{x}_\rho(t) |^2] \leq Ce^{Ct}I_\rho \leq Ce^{CT}I_\rho. \tag{4.2}$$

Note that $C_\rho(t)$ and $D_\rho(t)$ are bounded by $C'(| \hat{x}(t) | + | u(t) - \bar{u}(t) |)$, where C' is a constant which is independent with ρ. Thus, $\hat{\mathbb{E}}[| \tilde{x}_\rho(t) |^2]$ is bounded by some constant C for $0 \leq \rho \leq 1$.

(ii) By (4.2), we only need to prove that $I_\rho \to 0$ as $\rho \to 0$. We first prove

$$\lim_{\rho \to 0}\hat{\mathbb{E}}[\int_0^T | C_\rho(s) |^2 \, ds] = 0.$$

Define

$$E_\rho(t) = b_x(t, \bar{x}(t) + \lambda\rho(\hat{x}(t) + \tilde{x}_\rho(t)), \bar{u}(t) + \lambda\rho(u(t) - \bar{u}(t))) - b_x(t),$$
$$F_\rho(t) = b_u(t, \bar{x}(t) + \lambda\rho(\hat{x}(t) + \tilde{x}_\rho(t)), \bar{u}(t) + \lambda\rho(u(t) - \bar{u}(t))) - b_u(t).$$

For $N > 0$, set

$$S_{1,N} = \{| \hat{x}(t) + \tilde{x}_\rho(t) |\leq N\},$$
$$S_{2,N} = \{| u(t) - \bar{u}(t) |\leq N\}.$$

9

We have

$$
\begin{aligned}
\mid C_\rho(t) \mid^2 = \ & \mid \int_0^1 E_\rho(t)d\lambda \hat{x}(t) + \int_0^1 F_\rho(t)d\lambda(u(t) - \bar{u}(t)) \mid^2 \\
\leq \ & 2(\int_0^1 \mid E_\rho(t) \mid^2 d\lambda \mid \hat{x}(t) \mid^2 + \int_0^1 \mid F_\rho(t) \mid^2 d\lambda \mid u(t) - \bar{u}(t) \mid^2) \\
\leq \ & 2(\int_0^1 \mid E_\rho(t) \mid^2 (I_{S_{1,N} \cap S_{2,N}} + I_{S_{1,N}^c} + I_{S_{2,N}^c})d\lambda \mid \hat{x}(t) \mid^2 \\
& + \int_0^1 \mid F_\rho(t) \mid^2 (I_{S_{1,N} \cap S_{2,N}} + I_{S_{1,N}^c} + I_{S_{2,N}^c})d\lambda \mid u(t) - \bar{u}(t) \mid^2) \\
\leq \ & 2\bar{\omega}(2N\rho) \mid \hat{x}(t) \mid^2 + C(I_{S_{1,N}^c} + I_{S_{2,N}^c}) \mid \hat{x}(t) \mid^2 \\
& + 2\bar{\omega}(2N\rho) \mid u(t) - \bar{u}(t) \mid^2 + C(I_{S_{1,N}^c} + I_{S_{2,N}^c}) \mid u(t) - \bar{u}(t) \mid^2 .
\end{aligned}
\tag{4.3}
$$

By Lemma 4.2, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $A \in \mathcal{B}([0,T]) \times \mathcal{F}_T$ with $\hat{\mathbb{E}}[\int_0^T I_A(t, \omega)dt] < \delta$, we have that

$$
\hat{\mathbb{E}}[\int_0^T \mid \hat{x}(t) \mid^2 I_A dt] < \varepsilon,
$$

$$
\hat{\mathbb{E}}[\int_0^T \mid u(t) - \bar{u}(t) \mid^2 I_A dt] < \varepsilon.
$$

Note that

$$
\hat{\mathbb{E}}[\int_0^T (I_{S_{1,N}^c} + I_{S_{2,N}^c})dt] \leq \frac{1}{N^2}\hat{\mathbb{E}}[\int_0^T (\mid \hat{x}(t) + \tilde{x}_\rho(t) \mid^2 + \mid u(t) - \bar{u}(t) \mid^2)dt],
$$

then we can choose an $N > 0$ such that $\hat{\mathbb{E}}[\int_0^T (I_{S_{1,N}^c} + I_{S_{2,N}^c})dt] < \delta$, which implies that

$$
\hat{\mathbb{E}}[\int_0^T (I_{S_{1,N}^c} + I_{S_{2,N}^c})(\mid \hat{x}(t) \mid^2 + \mid u(t) - \bar{u}(t) \mid^2)dt] \leq C\varepsilon.
$$

Thus by (4.3), it is easy to obtain $\lim\limits_{\rho \to 0} \hat{\mathbb{E}}[\int_0^T \mid C_\rho(s) \mid^2 ds] = 0$. Similarly, we can prove that $\lim\limits_{\rho \to 0} \hat{\mathbb{E}}[\int_0^T \mid D_\rho(s) \mid^2 ds] = 0$. Thus we get $\lim\limits_{\rho \to 0} \sup\limits_{0 \leq t \leq T} \hat{\mathbb{E}}[\mid \tilde{x}_\rho(t) \mid^2] = 0$. $\square$

Now let

$$
f_\rho(t) = f(t, x_\rho(t), y_\rho(t), \bar{u}(t) + \rho(u(t) - \bar{u}(t))),\ f(t) = f(t, \bar{x}(t), \bar{y}(t), \bar{u}(t)),
$$

$$
f_x(t) = f_x(t, \bar{x}(t), \bar{y}(t), \bar{u}(t)),\ f_y(t) = f_y(t, \bar{x}(t), \bar{y}(t), \bar{u}(t)),\ f_u(t) = f_u(t, \bar{x}(t), \bar{y}(t), \bar{u}(t)).
$$

Set

$$
\mathcal{P}^* = \{P \in \mathcal{P} \mid E_P[\bar{K}(T)] = 0\}
$$

and

$$
\Theta^u = \phi_x(\bar{x}(T))\hat{x}(T)m(T) + \int_0^T [f_x(s)\hat{x}(s) + f_u(s)(u(s) - \bar{u}(s))]m(s)ds,
$$

where

$$
m(t) = \exp\{\int_0^t f_y(s)ds\}.
$$

Theorem 4.3 *Suppose (H1)-(H3) hold. Then, for any $u(\cdot) \in \mathcal{U}[0,T]$, there exists a $P^u \in \mathcal{P}^*$ such that*

$$
\lim_{\rho \to 0} \frac{y_\rho(0) - \bar{y}(0)}{\rho} = E_{P^u}[\Theta^u] = \sup_{P \in \mathcal{P}^*} E_P[\Theta^u].
\tag{4.4}
$$

Remark 4.4 *If B is the classical Brownian motion, then $E_{P^u}[\Theta^u]$ is the solution of the variational equation for BSDE at time 0.*

In order to prove this theorem, we need the following lemma.

Lemma 4.5 *Assume (H1)-(H3) hold. Then we have*

(i) $\hat{\mathbb{E}}[|\ J_1 m(T)\ |] = o(\rho)$,

(ii) $\hat{\mathbb{E}}[|\ \int_0^T f_x(s)\tilde{x}_\rho(s)m(s)ds\ |] = o(1)$,

(iii) $\hat{\mathbb{E}}[|\ \int_0^T J_2(s)m(s)ds\ |] = o(\rho)$,

where

$$J_1 = \phi(x_\rho(T)) - \phi(\bar{x}(T)) - \phi_x(\bar{x}(T))\rho\hat{x}(T),$$
$$J_2(s) = (f_\rho(s) - f(s)) - [f_x(s)(x_\rho(s) - \bar{x}(s)) + f_y(s)(y_\rho(s) - \bar{y}(s)) + f_u(s)\rho(u(s) - \bar{u}(s))].$$

Proof. (i)

$$
\begin{aligned}
J_1 =\ & \int_0^1 \phi_x(\bar{x}(T) + \lambda(x_\rho(T) - \bar{x}(T)))d\lambda(x_\rho(T) - \bar{x}(T)) - \phi_x(\bar{x}(T))\rho\hat{x}(T) \\
=\ & \int_0^1 \phi_x(\bar{x}(T) + \lambda\rho(\tilde{x}_\rho(T) + \hat{x}(T)))d\lambda\rho(\tilde{x}_\rho(T) + \hat{x}(T)) - \phi_x(\bar{x}(T))\rho\hat{x}(T) \\
=\ & \rho \int_0^1 (\phi_x(\bar{x}(T) + \lambda\rho(\tilde{x}_\rho(T) + \hat{x}(T))) - \phi_x(\bar{x}(T)))d\lambda\hat{x}(T) \\
& + \rho \int_0^1 \phi_x(\bar{x}(T) + \lambda\rho(\tilde{x}_\rho(T) + \hat{x}(T)))d\lambda\tilde{x}_\rho(T).
\end{aligned}
$$

Using the similar analysis as in Proposition 4.1, we can prove that

$$\lim_{\rho\to 0}\hat{\mathbb{E}}[|\ \int_0^1 (\phi_x(\bar{x}(T) + \lambda\rho(\tilde{x}_\rho(T) + \hat{x}(T))) - \phi_x(\bar{x}(T)))d\lambda\ ||\ \hat{x}(T)m(T)\ |] = 0.$$

It is easy to see

$$\hat{\mathbb{E}}[|\ \int_0^1 \phi_x(\bar{x}(T) + \lambda\rho(\tilde{x}_\rho(T) + \hat{x}(T)))d\lambda\tilde{x}_\rho(T)m(T)\ |] \le C(\hat{\mathbb{E}}[|\ \tilde{x}_\rho(T)\ |^2])^{\frac{1}{2}}(\hat{\mathbb{E}}[|\ m(T)\ |^2])^{\frac{1}{2}}.$$

Then, by Proposition 4.1,

$$\lim_{\rho\to 0}\hat{\mathbb{E}}[|\ \int_0^1 \phi_x(\bar{x}(T) + \lambda\rho(\tilde{x}_\rho(T) + \hat{x}(T)))d\lambda\tilde{x}_\rho(T)m(T)\ |] = 0.$$

(ii)

$$
\begin{aligned}
\hat{\mathbb{E}}[|\ \int_0^T f_x(s)\tilde{x}_\rho(s)m(s)ds\ |]\ & \le C\int_0^T \hat{\mathbb{E}}[|\ \tilde{x}_\rho(s)\ ||\ m(s)\ |]ds \\
& \le C\int_0^T (\hat{\mathbb{E}}[|\ \tilde{x}_\rho(s)\ |^2])^{\frac{1}{2}}ds \\
& \le CT(\sup_{0\le s\le T}\hat{\mathbb{E}}[|\ \tilde{x}_\rho(s)\ |^2])^{\frac{1}{2}}.
\end{aligned}
$$

By Proposition 4.1, $\hat{\mathbb{E}}[|\ \int_0^T f_x(s)\tilde{x}_\rho(s)m(s)ds\ |] \to 0$ as $\rho \to 0$.

(iii) Set

$$\tilde{f}_l(s) = f_l(s, \bar{x}(s) + \lambda\rho(\tilde{x}_\rho(s) + \hat{x}(s)), \bar{y}(s) + \lambda(y_\rho(s) - \bar{y}(s)), \bar{u}(s) + \lambda\rho(u(s) - \bar{u}(s)))$$

for $l = x, y, u$. Then

$$
\begin{aligned}
J_2(s) = & \int_0^1 (\tilde{f}_x(s) - f_x(s))d\lambda\rho(\tilde{x}_\rho(s) + \hat{x}(s)) + \int_0^1 (\tilde{f}_y(s) - f_y(s))d\lambda(y_\rho(s) - \bar{y}(s)) \\
& + \int_0^1 (\tilde{f}_u(s) - f_u(s))d\lambda\rho(u(s) - \bar{u}(s)).
\end{aligned}
$$

We only prove that

$$\hat{\mathbb{E}}[| \int_0^T \int_0^1 (\tilde{f}_y(s) - f_y(s))d\lambda(y_\rho(s) - \bar{y}(s))m(s)ds |] = o(\rho).$$

The proofs of the other terms are similar.

By Proposition 2.15 in [16], we have

$$| y_\rho(t) - \bar{y}(t) |^2$$

$$\leq C(\hat{\mathbb{E}}_t[| \phi(x_\rho(T)) - \phi(\bar{x}(T)) |^2]$$

$$+ \hat{\mathbb{E}}_t[\int_t^T | f(s, x_\rho(s), \bar{y}(s), \bar{u}(s) + \rho(u(s) - \bar{u}(s))) - f(s) |^2 ds]).$$

Then, by Proposition 4.1,

$$\sup_{0 \leq t \leq T} \hat{\mathbb{E}}[| y_\rho(t) - \bar{y}(t) |^2]$$

$$\leq C(\hat{\mathbb{E}}[| x_\rho(T) - \bar{x}(T) |^2]$$

$$+ \int_0^T (\hat{\mathbb{E}}[| x_\rho(t) - \bar{x}(t) |^2] + \hat{\mathbb{E}}[\rho^2 | u(t) - \bar{u}(t) |^2])dt)$$

$$\leq C(\hat{\mathbb{E}}[\rho^2 | \tilde{x}_\rho(t) + \hat{x}(t) |^2]$$

$$+ \int_0^T (\hat{\mathbb{E}}[\rho^2 | \tilde{x}_\rho(t) + \hat{x}(t) |^2] + \hat{\mathbb{E}}[\rho^2 | u(t) - \bar{u}(t) |^2])dt)$$

$$\leq C\rho^2.$$

Let $\alpha \in (0, 1)$ be fixed. For each $N > 0$, we have

$$\hat{\mathbb{E}}[| \int_0^T \int_0^1 (\tilde{f}_y(s) - f_y(s))I_{\{|y_\rho(s) - \bar{y}(s)| > N\rho\}}d\lambda(y_\rho(s) - \bar{y}(s))m(s)ds|]$$

$$\leq C\hat{\mathbb{E}}[\int_0^T I_{\{|y_\rho(s) - \bar{y}(s)| > N\rho\}} | y_\rho(s) - \bar{y}(s) | m(s)ds]$$

$$\leq \frac{C}{N^\alpha \rho^\alpha}\hat{\mathbb{E}}[\int_0^T | y_\rho(s) - \bar{y}(s) |^{1+\alpha} m(s)ds]$$

$$\leq \frac{C}{N^\alpha \rho^\alpha}(\hat{\mathbb{E}}[\int_0^T | y_\rho(s) - \bar{y}(s) |^2 ds])^{\frac{1+\alpha}{2}} (\hat{\mathbb{E}}[\int_0^T |m(s)|^{\frac{2}{1-\alpha}} ds])^{\frac{1-\alpha}{2}}$$

$$\leq \frac{C}{N^\alpha}\rho,$$

$$\hat{\mathbb{E}}[|\int_0^T \int_0^1 (\tilde{f}_y(s) - f_y(s))(I_{\{|\tilde{x}_\rho(s)+\hat{x}(s)|>N\}} + I_{\{|u(s)-\bar{u}(s)|>N\}})d\lambda(y_\rho(s) - \bar{y}(s))m(s)ds|]$$

$$\leq C\hat{\mathbb{E}}[\int_0^T (I_{\{|\tilde{x}_\rho(s)+\hat{x}(s)|>N\}} + I_{\{|u(s)-\bar{u}(s)|>N\}}) \mid y_\rho(s) - \bar{y}(s) \mid m(s)ds]$$

$$\leq \frac{C}{N^\alpha}\hat{\mathbb{E}}[\int_0^T (|\tilde{x}_\rho(s) + \hat{x}(s)|^\alpha + |u(s) - \bar{u}(s)|^\alpha) \mid y_\rho(s) - \bar{y}(s) \mid m(s)ds]$$

$$\leq \frac{C}{N^\alpha}(\hat{\mathbb{E}}[\int_0^T (|\tilde{x}_\rho(s) + \hat{x}(s)|^2 + |u(s) - \bar{u}(s)|^2)ds])^{\frac{\alpha}{2}}(\hat{\mathbb{E}}[\mid y_\rho(s) - \bar{y}(s) \mid^2])^{\frac{1}{2}}(\hat{\mathbb{E}}[\mid m(s) \mid^{\frac{2}{1-\alpha}}])^{\frac{1-\alpha}{2}}$$

$$\leq \frac{C}{N^\alpha}\rho$$

and

$$\hat{\mathbb{E}}[|\int_0^T \int_0^1 (\tilde{f}_y(s) - f_y(s))I_{\{|\tilde{x}_\rho(s)+\hat{x}(s)|\leq N\}\cap\{|y_\rho(s)-\bar{y}(s)|\leq N\rho\}\cap\{|u(s)-\bar{u}(s)|\leq N\}}d\lambda(y_\rho(s) - \bar{y}(s))m(s)ds|]$$

$$\leq C\hat{\mathbb{E}}[\int_0^T \bar{\omega}(3N\rho) \mid y_\rho(s) - \bar{y}(s) \mid m(s)ds]$$

$$\leq C\bar{\omega}(3N\rho)(\hat{\mathbb{E}}[\int_0^T \mid y_\rho(s) - \bar{y}(s) \mid^2 ds])^{\frac{1}{2}}(\hat{\mathbb{E}}[\int_0^T |m(s)|^2 ds])^{\frac{1}{2}}$$

$$\leq C\bar{\omega}(3N\rho)\rho.$$

Thus we get for each $N > 0$,

$$\hat{\mathbb{E}}[\mid \int_0^T \int_0^1 (\tilde{f}_y(s) - f_y(s))d\lambda(y_\rho(s) - \bar{y}(s))m(s)ds \mid]$$
$$\leq C\bar{\omega}(3N\rho)\rho + \frac{C}{N^\alpha}\rho,$$

which easily implies that $\hat{\mathbb{E}}[\mid \int_0^T \int_0^1 (\tilde{f}_y(s) - f_y(s))d\lambda(y_\rho(s) - \bar{y}(s))m(s)ds \mid] = o(\rho)$.

The proof is complete. ∎

Proof of Theorem 4.3.

Step 1. We first prove that $\lim_{\rho\to 0}\frac{y_\rho(0)-\bar{y}(0)}{\rho}$ exists.

Consider

$$y_\rho(t) - \bar{y}(t) = \phi(x_\rho(T)) - \phi(\bar{x}(T)) + \int_t^T (f_\rho(s) - f(s))ds - \int_t^T (z_\rho(s) - \bar{z}(s))dB(s)$$
$$-(K_\rho(T) - K_\rho(t)) + (\bar{K}(T) - \bar{K}(t)).$$

It yields that

$$\bar{K}(t) + y_\rho(t) - \bar{y}(t) = \bar{K}(T) + \phi_x(\bar{x}(T))\rho\hat{x}(T) + J_1 - \int_t^T (z_\rho(s) - \bar{z}(s))dB(s) - (K_\rho(T) - K_\rho(t))$$
$$+ \int_t^T [f_x(s)(x_\rho(s) - \bar{x}(s)) + f_y(s)(y_\rho(s) - \bar{y}(s)) + f_u(s)\rho(u(s) - \bar{u}(s)) + J_2(s)]ds.$$

Applying Itô's formula to $m(t)(\bar{K}(t) + y_\rho(t) - \bar{y}(t))$, we can get

$$y_\rho(0) - \bar{y}(0) = \hat{\mathbb{E}}[(\bar{K}(T) + \phi_x(\bar{x}(T))\rho\hat{x}(T) + J_1)m(T)$$
$$+ \int_t^T (f_x(s)(x_\rho(s) - \bar{x}(s)) - f_y(s)\bar{K}(s) + f_u(s)\rho(u(s) - \bar{u}(s)) + J_2(s))m(s)ds]. \tag{4.5}$$

Note that

$$\bar{K}(T)m(T) = \int_0^T f_y(s)\bar{K}(s)m(s)ds + \int_0^T m(s)d\bar{K}(s),$$

13

then (4.5) becomes

$$y_\rho(0) - \bar{y}(0) = \hat{\mathbb{E}}[(\phi_x(\bar{x}(T))\rho\hat{x}(T) + J_1)m(T) + \int_0^T m(s)d\bar{K}(s)$$
$$+ \int_t^T (f_x(s)(x_\rho(s) - \bar{x}(s)) + f_u(s)\rho(u(s) - \bar{u}(s)) + J_2(s))m(s)ds].$$

By Lemma 4.5,

$$y_\rho(0) - \bar{y}(0) = \hat{\mathbb{E}}[\phi_x(\bar{x}(T))\rho\hat{x}(T)m(T) + \int_0^T m(s)d\bar{K}(s)$$
$$+ \int_0^T (f_x(s)\rho\hat{x}(s) + f_u(s)\rho(u(s) - \bar{u}(s)))m(s)ds] + o(\rho). \tag{4.6}$$

Since $\bar{u}(\cdot)$ is an optimal control, we have

$$\frac{y_\rho(0) - \bar{y}(0)}{\rho} = \hat{\mathbb{E}}[\frac{\int_0^T m(s)d\bar{K}(s)}{\rho} + \Theta^u] + o(1) \geq 0. \tag{4.7}$$

Note that $\frac{\int_0^T m(s)d\bar{K}(s)}{\rho}$ decreases as $\rho \downarrow 0$. It yields that $\hat{\mathbb{E}}[\frac{\int_0^T m(s)d\bar{K}(s)}{\rho} + \Theta^u]$ decreases. Since $\hat{\mathbb{E}}[\cdot]$ is sublinear,

$$\hat{\mathbb{E}}[\frac{\int_0^T m(s)d\bar{K}(s)}{\rho} + \Theta^u] \geq \hat{\mathbb{E}}[\frac{\int_0^T m(s)d\bar{K}(s)}{\rho}] - \hat{\mathbb{E}}[-\Theta^u] = -\hat{\mathbb{E}}[-\Theta^u].$$

Thus, the limit of $\frac{y_\rho(0) - \bar{y}(0)}{\rho}$ exists as $\rho \to 0$.

Step 2. Then, we prove that there exists a $P^u \in \mathcal{P}$ such that $E_{P^u}[\bar{K}(T)] = 0$.

Since $\mathcal{P}$ is weakly compact and $\int_0^T m(s)d\bar{K}(s) + \rho\Theta^u \in L^2_G(\Omega_T)$, there exists a $P^{\rho,u} \in \mathcal{P}$ which depends on ρ and $u(\cdot)$ such that

$$\hat{\mathbb{E}}[\frac{\int_0^T m(s)d\bar{K}(s)}{\rho} + \Theta^u] = E_{P^{\rho,u}}[\frac{\int_0^T m(s)d\bar{K}(s)}{\rho} + \Theta^u].$$

Thus (4.7) becomes

$$\frac{y_\rho(0) - \bar{y}(0)}{\rho} = E_{P^{\rho,u}}[\frac{\int_0^T m(s)d\bar{K}(s)}{\rho} + \Theta^u] + o(1) \geq 0. \tag{4.8}$$

Obviously, there exist a $P^u \in \mathcal{P}$ and a sequence $P^{\rho_n,u} \to P^u$ weakly as $\rho_n \to 0$. By (4.8), we get

$$E_{P^{\rho_n,u}}[\int_0^T m(s)d\bar{K}(s)] = y_{\rho_n}(0) - \bar{y}(0) - \rho_n E_{P^{\rho_n,u}}[\Theta^u] - o(\rho_n).$$

Note that

$$| E_{P^{\rho_n,u}}[\Theta^u] | \leq \hat{\mathbb{E}}[|\Theta^u|] < \infty,$$

it yields that $E_{P^{\rho_n,u}}[\int_0^T m(s)d\bar{K}(s)] \to 0$ as $n \to \infty$. Since $\int_0^T m(s)d\bar{K}(s)$ belongs to $L^2_G(\Omega_T)$, it is easy to see that

$$E_{P^{\rho_n,u}}[\int_0^T m(s)d\bar{K}(s)] \to E_{P^u}[\int_0^T m(s)d\bar{K}(s)].$$

Thus we deduce that $E_{P^u}[\bar{K}(T)] = 0$.

Step 3. At last, we prove that $\lim_{\rho\to 0} \frac{y_\rho(0) - \bar{y}(0)}{\rho} = E_{P^u}[\Theta^u] = \sup_{P\in\mathcal{P}^*} E_P[\Theta^u]$.

14

By (4.8) and $\int_0^T m(s)d\bar{K}(s) \leq 0$,

$$\frac{y_\rho(0) - \bar{y}(0)}{\rho} \leq E_{P^{\rho,u}}[\Theta^u] + o(1).$$

Then

$$\lim_{\rho \to 0} \frac{y_\rho(0) - \bar{y}(0)}{\rho} \leq \lim_{n \to \infty} E_{P^{\rho_n,u}}[\Theta^u] = E_{P^u}[\Theta^u]. \tag{4.9}$$

For any $P \in \mathcal{P}^*$, by (4.7), we have

$$\begin{aligned}
\frac{y_\rho(0) - \bar{y}(0)}{\rho} &= \hat{\mathbb{E}}\left[\frac{\int_0^T m(s)d\bar{K}(s)}{\rho} + \Theta^u\right] + o(1) \\
&\geq E_P\left[\frac{\int_0^T m(s)d\bar{K}(s)}{\rho} + \Theta^u\right] + o(1) \\
&= E_P[\Theta^u] + o(1).
\end{aligned}$$

It yields that

$$\lim_{\rho \to 0} \frac{y_\rho(0) - \bar{y}(0)}{\rho} \geq E_P[\Theta^u], \ \forall P \in \mathcal{P}^*. \tag{4.10}$$

Note that $P^u \in \mathcal{P}^*$, then, by (4.9) and (4.10), we obtain

$$\lim_{\rho \to 0} \frac{y_\rho(0) - \bar{y}(0)}{\rho} = E_{P^u}[\Theta^u] = \sup_{P \in \mathcal{P}^*} E_P[\Theta^u].$$

This completes the proof. $\square$

4.2 Variational inequality

We obtain the following variational inequality.

Theorem 4.6 *Suppose (H1)-(H3) hold. Then there exists a $P^* \in \mathcal{P}^*$ such that*

$$\inf_{u \in \mathcal{U}[0,T]} E_{P^*}[\Theta^u] \geq 0.$$

Proof. By Theorem 4.3, we can get for any $u(\cdot) \in \mathcal{U}[0,T]$,

$$\lim_{\rho \to 0} \frac{y_\rho(0) - \bar{y}(0)}{\rho} = \sup_{P \in \mathcal{P}^*} E_P[\Theta^u] \geq 0.$$

Then

$$\inf_{u \in \mathcal{U}[0,T]} \sup_{P \in \mathcal{P}^*} E_P[\Theta^u] \geq 0.$$

It is easy to check that $\mathcal{P}^*$ is convex and weakly compact, and for $\lambda \in [0,1]$, u, $u' \in \mathcal{U}[0,T]$,

$$\Theta^{\lambda u + (1-\lambda)u'} = \lambda \Theta^u + (1-\lambda)\Theta^{u'}.$$

Thus, by Sion's minimax theorem, we obtain

$$\inf_{u \in \mathcal{U}[0,T]} \sup_{P \in \mathcal{P}^*} E_P[\Theta^u] = \sup_{P \in \mathcal{P}^*} \inf_{u \in \mathcal{U}[0,T]} E_P[\Theta^u].$$

Then, for each $\varepsilon > 0$, there exists a $P^\varepsilon \in \mathcal{P}^*$ such that

$$\inf_{u \in \mathcal{U}[0,T]} E_{P^\varepsilon}[\Theta^u] \geq -\varepsilon.$$

Since $\mathcal{P}^*$ is weakly compact, there exist a $P^* \in \mathcal{P}^*$ and a sequence $P^{\varepsilon_n} \to P^*$ weakly as $\varepsilon_n \to 0$. Note that for any $u(\cdot) \in \mathcal{U}[0,T]$,

$$E_{P^{\varepsilon_n}}[\Theta^u] \geq -\varepsilon_n.$$

Letting $\varepsilon_n \to 0$, it yields that for any $u(\cdot) \in \mathcal{U}[0,T]$,

$$E_{P^*}[\Theta^u] \geq 0.$$

Thus, we have

$$\inf_{u \in \mathcal{U}[0,T]} E_{P^*}[\Theta^u] \geq 0.$$

This completes the proof. ∎

4.3 Maximum principle

Consider the following kind of BSDE under P^*:

$$\begin{cases} -dp(t) = & [(f_x(t))^T + (b_x(t))^T p(t) + f_y(t)p(t)]dt + (\sigma_x^i(t))^T q^j(t)d\langle B^i, B^j\rangle(t) - q^i(t)dB^i(t) - dN(t), \\ p(T) = & (\phi_x(\bar{x}(T)))^T, \end{cases}$$

$$(4.11)$$

where $(p(t))_{t \in [0,T]} \in M_{P^*}^2(0,T;\mathbb{R}^n) = \{\eta : \eta$ is $\mathbb{R}^n$-valued progressively measurable and $E_{P^*}[\int_0^T |\eta_t|^2 dt] < \infty\}$, $(q(t))_{t \in [0,T]} \in M_{P^*}^2(0,T;\mathbb{R}^{n \times d})$, $(N_t)_{t \in [0,T]} \in \mathcal{M}_{P^*}^{2,\perp}(0,T;\mathbb{R}^n) := \{N :$ all $\mathbb{R}^n$-valued square integrable martingale that is orthogonal to $B\}$.

Remark 4.7 *Note that B is only a continuous martingale under P^* and the martingale representation theorem may not hold. So it is necessary to introduce the third term N which is orthogonal to B.*

Following El Karoui and Huang [7] and Buckdahn et. al. [2], there exists a unique $(p(\cdot), q(\cdot), N(\cdot)) \in M_{P^*}^2(0,T;\mathbb{R}^n) \times M_{P^*}^2(0,T;\mathbb{R}^{n \times d}) \times \mathcal{M}_{P^*}^{2,\perp}(0,T;\mathbb{R}^n)$ which solves the adjoint equation (4.11). Applying Itô's formula to $\langle \hat{x}(t), m(t)p(t) \rangle$, we obtain

$$E_{P^*}[\phi_x(\bar{x}(T))\hat{x}(T)m(T) + \int_0^T f_x(s)\hat{x}(s)m(s)ds]$$

$$= E_{P^*}[\int_0^T (m(t)\langle p(t), b_u(t)(u(t) - \bar{u}(t))\rangle + m(t)\langle q^j(t), \sigma_u^i(t)(u(t) - \bar{u}(t))\rangle\gamma^{ij}(t))dt],$$

where $\Gamma(t) = (\gamma^{ij}(t))$, $d\langle B^i, B^j\rangle(t) = \gamma^{ij}(t)dt$. We define the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times [0,T] \to \mathbb{R}$ as follows:

$$H(x, y, u, p, q, t) = \langle p, b(t, x, u)\rangle + \langle q^j, \sigma^i(t, x, u)\rangle\gamma^{ij}(t) + f(t, x, y, u).$$

Thus

$$
\begin{aligned}
E_{P^*}[\Theta^u] &= E_{P^*}[\int_0^T m(t)\langle (b_u(t))^T p(t) + (f_u(t))^T + (\sigma_u^i(t))^T q^j(t)\gamma^{ij}(t), u(t) - \bar{u}(t)\rangle dt] \\
&= E_{P^*}[\int_0^T m(t)\langle (H_u(\bar{x}(t), \bar{y}(t), \bar{u}(t), p(t), q(t), t))^T, u(t) - \bar{u}(t)\rangle dt] \\
&= E_{P^*}[\int_0^T m(t) H_u(\bar{x}(t), \bar{y}(t), \bar{u}(t), p(t), q(t), t)(u(t) - \bar{u}(t)) dt].
\end{aligned}
$$

By Theorem 4.6, $E_{P^*}[\Theta^u] \geq 0$ for each $u(\cdot) \in \mathcal{U}[0, T]$, then we can get

$$
H_u(\bar{x}(t), \bar{y}(t), \bar{u}(t), p(t), q(t), t)(u - \bar{u}(t)) \geq 0, \ \forall u \in U, \ a.e., \ P^* - a.s.. \tag{4.12}
$$

We summarize the above analysis to the following stochastic maximum principle.

Theorem 4.8 *Suppose (H1)-(H3) hold. Let $\bar{u}(\cdot)$ be an optimal control and $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot), \bar{K}(\cdot))$ be the corresponding trajectory. Then there exist a $P^* \in \mathcal{P}^*$ and $(p(\cdot), q(\cdot), N(\cdot)) \in M_{P^*}^2(0, T; \mathbb{R}^n) \times M_{P^*}^2(0, T; \mathbb{R}^{n \times d}) \times \mathcal{M}_{P^*}^{2,\perp}(0, T; \mathbb{R}^n)$, which is the solution of the adjoint equation (4.11), such that the inequality (4.12) holds.*

4.4 Sufficient condition

In this subsection, we give the sufficient condition for optimality.

Theorem 4.9 *Suppose (H1)-(H3) hold. Let $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ and $P^* \in \mathcal{P}^*$ satisfy that*

$$
H_u(\bar{x}(t), \bar{y}(t), \bar{u}(t), p(t), q(t), t)(u - \bar{u}(t)) \geq 0, \ \forall u \in U, \ a.e., \ P^* - a.s.,
$$

where $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot), \bar{K}(\cdot))$ is the state processes of (3.1) and (3.2) corresponding to $\bar{u}(\cdot)$ and $(p(\cdot), q(\cdot), N(\cdot))$ is the solution of the adjoint equation (4.11) under P^*. We also assume that H is convex with respect to x, y, u and ϕ is convex with respect to x. Then $\bar{u}(\cdot)$ is an optimal control.

Proof. For any $u(\cdot) \in \mathcal{U}[0, T]$, let $(x(\cdot), y(\cdot), z(\cdot), K(\cdot))$ be the corresponding state processes of (3.1) and (3.2). Define $\xi(t) := x(t) - \bar{x}(t)$ and $\eta(t) := y(t) - \bar{y}(t)$. Then $\xi(\cdot)$ and $\eta(\cdot)$ satisfy the following equations under P^*:

$$
\begin{cases}
d\xi(t) = & [b_x(t)\xi(t) + \alpha(t)]dt + [\sigma_x^i(t)\xi(t) + \beta^i(t)]dB^i(t), \\
\xi(0) = & 0,
\end{cases}
$$

where

$$
\begin{aligned}
\alpha(t) &:= -b_x(t)\xi(t) + b(t, x(t), u(t)) - b(t, \bar{x}(t), \bar{u}(t)), \\
\beta^i(t) &:= -\sigma_x^i(t)\xi(t) + \sigma^i(t, x(t), u(t)) - \sigma^i(t, \bar{x}(t), \bar{u}(t)),
\end{aligned}
$$

and

$$
\begin{cases}
-d\eta(t) = & [f_x(t)\xi(t) + f_y(t)\eta(t) + \tilde{\alpha}(t)]dt - \tilde{z}(t)dB(t) - dK(t), \\
\eta(T) = & \phi_x(\bar{x}(T))\xi(T) + \tilde{\beta}(T),
\end{cases}
$$

17

where $\tilde{z}(t) := z(t) - \bar{z}(t)$,

$$\tilde{\alpha}(t) := -f_x(t)\xi(t) - f_y(t)\eta(t) + f(t, x(t), y(t), u(t)) - f(t, \bar{x}(t), \bar{y}(t), \bar{u}(t)),$$
$$\tilde{\beta}(T) := -\phi_x(\bar{x}(T))\xi(T) + \phi(x(T)) - \phi(\bar{x}(T)).$$

For simplicity, set

$$H(t) := H(\bar{x}(t), \bar{y}(t), \bar{u}(t), p(t), q(t), t).$$

The definitions of $H_x(t)$, $H_y(t)$ and $H_u(t)$ are similar. Applying Itô's lemma to $\langle \xi(t), m(t)p(t) \rangle - \eta(t)m(t)$ under P^*, we can derive

$$E_{P^*}[\langle \xi(T), m(T)p(T) \rangle - \langle \xi(0), m(0)p(0) \rangle - \eta(T)m(T) + \eta(0)m(0)]$$

$$= E_{P^*}[\int_0^T (\langle m(t)p(t), \alpha(t) \rangle + \langle m(t)q^j(t), \beta^i(t) \rangle \gamma^{ij}(t) + m(t)\tilde{\alpha}(t))dt - \int_0^T m(t)dK(t)]$$

$$= E_{P^*}[\int_0^T (-H_x(t)\xi(t) - H_y(t)\eta(t) + H(x(t), y(t), u(t), p(t), q(t), t) - H(t))m(t)dt - \int_0^T m(t)dK(t)]$$

$$\geq E_{P^*} \int_0^T [-H_x(t)\xi(t) - H_y(t)\eta(t) - H_u(t)(u(t) - \bar{u}(t)) + H(x(t), y(t), u(t), p(t), q(t), t) - H(t)]m(t)dt.$$

The last inequality is due to the assumption and $-m(t)dK(t) \geq 0$. Note that H is convex with respect to x, y, u. We have

$$-H_x(t)\xi(t) - H_y(t)\eta(t) - H_u(t)(u(t) - \bar{u}(t)) \geq H(t) - H(x(t), y(t), u(t), p(t), q(t), m(t), t).$$

It yields that

$$E_{P^*}[\langle \xi(T), m(T)p(T) \rangle - \langle \xi(0), m(0)p(0) \rangle - \eta(T)m(T) + \eta(0)m(0)] \geq 0,$$

which leads to $E_{P^*}[-\tilde{\beta}(T)m(T) + \eta(0)] \geq 0$. Since ϕ is convex with respect to x, we have that $\tilde{\beta}(T) \geq 0$. Thus, $\eta(0) \geq 0$, which implies that $\bar{u}(\cdot)$ is an optimal control. This completes the proof. ∎

5 The general case

In this section, we consider the general state equations.

5.1 f includes z term

Now we study the case in which the generator f of (3.2) includes the term z and we use the notations in Section 4. For simplicity, we assume that f only contains the term z, the other terms can be analyzed similarly as in Section 4. Similar to the proof of Theorem **4.3**, we can get

$$\bar{K}(t) + y_\rho(t) - \bar{y}(t) = \quad \bar{K}(T) + \phi_x(\bar{x}(T))\rho\hat{x}(T) + J_1 + \int_t^T A_\rho(s)(z_\rho(s) - \bar{z}(s))^T ds$$
$$- \int_t^T (z_\rho(s) - \bar{z}(s))dB(s) - (K_\rho(T) - K_\rho(t)),$$

where J_1 is the same as in Section 4 and $A_\rho(s) := \int_0^1 f_z(\bar{z}(s) + \lambda(z_\rho(s) - \bar{z}(s)))d\lambda$. Following [15], we construct an auxiliary extended $\tilde{G}$-expectation space $(\tilde{\Omega}, L_{\tilde{G}}^1(\tilde{\Omega}), \hat{\mathbb{E}}^{\tilde{G}})$ with $\tilde{\Omega} = C_0([0, \infty), \mathbb{R}^{2d})$ and

$$\tilde{G}(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \operatorname{tr} \left[A \begin{bmatrix} \gamma \gamma^T & I \\ I & (\gamma \gamma^T)^{-1} \end{bmatrix} \right], \ A \in \mathbb{S}_{2d}.$$

Let $(B(t), \tilde{B}(t))_{t \geq 0}$ be the canonical process in the extended space. It is easy to check that $\langle B^i, \tilde{B}^j \rangle(t) = \delta_{ij} t$. Consider the equation

$$dm_\rho(t) = A_\rho(t) m_\rho(t) d\tilde{B}(t), \ m_\rho(0) = 1.$$

Applying Itô's formula to $m_\rho(t)(\bar{K}(t) + y_\rho(t) - \bar{y}(t))$, we can get

$$y_\rho(0) - \bar{y}(0) = \hat{\mathbb{E}}^{\tilde{G}}[(\bar{K}(T) + \phi_x(\bar{x}(T))\rho \hat{x}(T) + J_1) m_\rho(T)]. \tag{5.1}$$

Note that

$$\bar{K}(T) m_\rho(T) = \int_0^T A_\rho(s) \bar{K}(s) m_\rho(s) d\tilde{B}(s) + \int_0^T m_\rho(s) d\bar{K}(s),$$

then (5.1) becomes

$$\begin{aligned} y_\rho(0) - \bar{y}(0) &= \hat{\mathbb{E}}^{\tilde{G}}[(\phi_x(\bar{x}(T))\rho \hat{x}(T) + J_1) m_\rho(T) + \int_0^T m_\rho(s) d\bar{K}(s)] \\ &= \hat{\mathbb{E}}^{\tilde{G}}[(\phi_x(\bar{x}(T))\rho \hat{x}(T) m(T) + \int_0^T m_\rho(s) d\bar{K}(s) + J_1 m_\rho(T) + J_2], \end{aligned}$$

where $J_2 = \phi_x(\bar{x}(T))\rho \hat{x}(T)(m_\rho(T) - m(T))$ and

$$dm(t) = f_z(t) m(t) d\tilde{B}(t), \ m(0) = 1.$$

Similar to the proof of Lemma 4.5, we can obtain $\hat{\mathbb{E}}^{\tilde{G}}[\| J_1 m_\rho(T) \|] = o(\rho)$. By Proposition 3.8 in [16], we can get

$$\hat{\mathbb{E}}^{\tilde{G}}[\int_0^T |z_\rho(s) - \bar{z}(s)|^2 ds] \leq C \rho.$$

Then similar to the proof of Proposition 4.1, we can easily obtain $\hat{\mathbb{E}}^{\tilde{G}}[|\hat{x}(T)(m_\rho(T) - m(T))|] = o(1)$. Thus we get

$$\frac{y_\rho(0) - \bar{y}(0)}{\rho} = \hat{\mathbb{E}}^{\tilde{G}}[\frac{\int_0^T m_\rho(s) d\bar{K}(s)}{\rho} + \phi_x(\bar{x}(T))\hat{x}(T) m(T)] + o(1).$$

We can choose a sequence $\rho_k \downarrow 0$ such that $\tilde{P}^{k,u} \in \tilde{\mathcal{P}}$ converges weakly to $\tilde{P}^u \in \tilde{\mathcal{P}}$ and

$$\lim_{k \to \infty} \frac{y_{\rho_k}(0) - \bar{y}(0)}{\rho_k} = \limsup_{\rho \to 0} \frac{y_\rho(0) - \bar{y}(0)}{\rho},$$

$$\hat{\mathbb{E}}^{\tilde{G}}[\frac{\int_0^T m_{\rho_k}(s) d\bar{K}(s)}{\rho_k} + \phi_x(\bar{x}(T))\hat{x}(T) m(T)] = E_{\tilde{P}^{k,u}}[\frac{\int_0^T m_{\rho_k}(s) d\bar{K}(s)}{\rho_k} + \phi_x(\bar{x}(T))\hat{x}(T) m(T)],$$

where $\tilde{\mathcal{P}}$ represents $\hat{\mathbb{E}}^{\tilde{G}}[\cdot]$. It is easy to check that $E_{\tilde{P}^{k,u}}[\int_0^T m_{\rho_k}(s) d\bar{K}(s)] \to 0$ as $k \to \infty$. Note that

$$\hat{\mathbb{E}}^{\tilde{G}}[| \int_0^T (m_\rho(s) - m(s)) d\bar{K}(s)|] \to 0 \text{ as } \rho \to 0,$$

then we can get $E_{\tilde{P}^{k,u}}[\int_0^T m(s)d\bar{K}(s)] \to 0$ as $k \to \infty$. Similar to the proof of Theorem **4.3**, we can get $\tilde{P}^u \in \tilde{\mathcal{P}}^* = \{\tilde{P} \in \tilde{\mathcal{P}} : E_{\tilde{P}}[\bar{K}(T)] = 0\}$ and

$$\sup_{\tilde{P} \in \tilde{\mathcal{P}}^*} E_{\tilde{P}}[\phi_x(\bar{x}(T))\hat{x}(T)m(T)] \leq \liminf_{\rho \to 0} \frac{y_\rho(0) - \bar{y}(0)}{\rho} \leq \limsup_{\rho \to 0} \frac{y_\rho(0) - \bar{y}(0)}{\rho} \leq E_{\tilde{P}^u}[\phi_x(\bar{x}(T))\hat{x}(T)m(T)],$$

which implies

$$\lim_{\rho \to 0} \frac{y_\rho(0) - \bar{y}(0)}{\rho} = E_{\tilde{P}^u}[\phi_x(\bar{x}(T))\hat{x}(T)m(T)].$$

Similar to the proof of Theorem 4.6, there exists a $\tilde{P}^* \in \tilde{\mathcal{P}}^*$ such that

$$\inf_{u \in \mathcal{U}[0,T]} E_{\tilde{P}^*}[\phi_x(\bar{x}(T))\hat{x}(T)m(T)] \geq 0.$$

Now we introduce the following adjoint equation under $\tilde{P}^*$:

$$
\begin{cases}
-d\tilde{p}(t) = & \{[(b_x(t))^T + f_{z_i}(t)(\sigma_x^i(t))^T]\tilde{p}(t) + f_{z_i}(t)\tilde{q}^{1,i}(t)\}dt \\
& + (\sigma_x^i(t))^T \tilde{q}^{1,j}(t)d\langle B^i, B^j\rangle(t) + f_{z_j}(t)[\tilde{q}^{2,i}(t) - f_{z_i}(t)\tilde{p}(t)]d\langle \tilde{B}^i, \tilde{B}^j\rangle(t) \\
& - \tilde{q}^{1,i}(t)dB^i(t) - [\tilde{q}^{2,i}(t) - f_{z_i}(t)\tilde{p}(t)]d\tilde{B}^i(t) - d\tilde{N}(t), \\
\tilde{p}(T) = & (\phi_x(\bar{x}(T)))^T.
\end{cases}
\tag{5.2}
$$

Set $\mathcal{F} = \sigma(B_t : t \geq 0)$ and $P^* = \tilde{P}^* \mid_{\mathcal{F}}$. We first show that $(\tilde{p}(\cdot), \tilde{q}^1(\cdot), \tilde{N}(\cdot)) \in M_{P^*}^2(0,T;\mathbb{R}^n) \times M_{P^*}^2(0,T;\mathbb{R}^{n\times d}) \times \mathcal{M}_{P^*}^{2,\perp}(0,T;\mathbb{R}^n)$. For this we consider the following BSDE under $(\Omega, \mathcal{F}, P^*)$:

$$
\begin{cases}
-dp(t) = & \{[(b_x(t))^T + f_{z_i}(t)(\sigma_x^i(t))^T]p(t) + f_{z_i}(t)q^i(t)\}dt \\
& + (\sigma_x^i(t))^T q^j(t)d\langle B^i, B^j\rangle(t) - q^i(t)dB^i(t) - dN(t), \\
p(T) = & (\phi_x(\bar{x}(T)))^T.
\end{cases}
\tag{5.3}
$$

By [2, 7], the above BSDE has a unique solution $(p(\cdot), q(\cdot), N(\cdot)) \in M_{P^*}^2(0,T;\mathbb{R}^n) \times M_{P^*}^2(0,T;\mathbb{R}^{n\times d}) \times \mathcal{M}_{P^*}^{2,\perp}(0,T;\mathbb{R}^n)$. It is easy to check that

$$(\tilde{p}(\cdot), \tilde{q}^1(\cdot), \tilde{q}^2(\cdot), \tilde{N}(\cdot)) = (p(\cdot), q(\cdot), p(\cdot)f_z(\cdot), N(\cdot)) \tag{5.4}$$

is the unique solution of the adjoint equation (5.2). Applying Itô's formula to $\langle \hat{x}(t), m(t)\tilde{p}(t)\rangle$ under $\tilde{P}^*$ and relation (5.4), we can get

$$E_{\tilde{P}^*}[\phi_x(\bar{x}(T))\hat{x}(T)m(T)]$$
$$= E_{\tilde{P}^*}[\int_0^T (\langle m(t)p(t), b_u(t)(u(t) - \bar{u}(t))\rangle + \langle m(t)f_{z_i}(t)p(t), \sigma_u^i(t)(u(t) - \bar{u}(t))\rangle$$
$$+ \langle m(t)q^j(t), \sigma_u^i(t)(u(t) - \bar{u}(t))\rangle\gamma^{ij}(t))dt].$$

We define the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1\times d} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times [0,T] \to \mathbb{R}$ as follows:

$$H(x, z, u, p, q, t) = \langle p, b(t, x, u)\rangle + \langle f_{z_i}(z)p, \sigma^i(t, x, u)\rangle + \langle q^j, \sigma^i(t, x, u)\rangle\gamma^{ij}(t) + f(z).$$

Then
$$E_{\tilde{P}^*}[\phi_x(\bar{x}(T))\hat{x}(T)m(T)] = E_{\tilde{P}^*}[\int_0^T m(t)H_u(\bar{x}(t),\bar{z}(t),\bar{u}(t),p(t),q(t),t)(u(t)-\bar{u}(t))dt].$$

Thus
$$H_u(\bar{x}(t),\bar{z}(t),\bar{u}(t),p(t),q(t),t)(u-\bar{u}(t)) \geq 0, \ \forall u \in U, \ a.e., \ \tilde{P}^* - a.s..$$

Note that all the terms in the above inequality are measurable with respect to $\mathcal{F}$, then we get
$$H_u(\bar{x}(t),\bar{z}(t),\bar{u}(t),p(t),q(t),t)(u-\bar{u}(t)) \geq 0, \ \forall u \in U, \ a.e., \ P^* - a.s.. \tag{5.5}$$

We summarize the above analysis to the following theorem.

Theorem 5.1 *Suppose (H1)-(H3) hold and f only depends on the term z. Let $\bar{u}(\cdot)$ be an optimal control and $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot), \bar{K}(\cdot))$ be the corresponding trajectory. Then there exist a $P^* \in \mathcal{P}^*$ and $(p(\cdot), q(\cdot), N(\cdot)) \in M_{P^*}^2(0,T;\mathbb{R}^n) \times M_{P^*}^2(0,T;\mathbb{R}^{n\times d}) \times \mathcal{M}_{P^*}^{2,\perp}(0,T;\mathbb{R}^n)$, which is the solution of the adjoint equation (5.3), such that the inequality (5.5) holds.*

5.2   The general maximum principle

In this subsection, we study the general case, i.e. the state equations are governed by (3.1) and (3.2). We only list the main results since the proofs are similar as in section 4 and subsection 5.1.

For this case, we introduce the following variational equation:

$$\begin{cases} d\hat{x}(t) = & [b_x(t)\hat{x}(t) + b_u(t)(u(t)-\bar{u}(t))]dt + [h_x^{ij}(t)\hat{x}(t) + h_u^{ij}(t)(u(t)-\bar{u}(t))]d\langle B^i, B^j\rangle(t) \\ & + [\sigma_x^i(t)\hat{x}(t) + \sigma_u^i(t)(u(t)-\bar{u}(t))]dB^i(t), \\ \hat{x}(0) = & 0. \end{cases}$$

Similarly, for some $P^* \in \mathcal{P}^*$, the following adjoint equation has a unique solution $(p(\cdot), q(\cdot), N(\cdot)) \in M_{P^*}^2(0,T;\mathbb{R}^n) \times M_{P^*}^2(0,T;\mathbb{R}^{n\times d}) \times \mathcal{M}_{P^*}^{2,\perp}(0,T;\mathbb{R}^n)$.

$$\begin{cases} -dp(t) = & \{(f_x(t))^T + [(b_x(t))^T + f_{z_i}(t)(\sigma_x^i(t))^T + f_y(t)]p(t) + f_{z_i}(t)q^l(t)\}dt \\ & + \{(g_x^{ij}(t))^T + [(h_x^{ij}(t))^T + g_{z_l}^{ij}(t)(\sigma_x^l(t))^T + g_y^{ij}(t)]p(t) + g_{z_l}^{ij}(t)q^l(t) \\ & (\sigma_x^i(t))^T q^j(t)\}d\langle B^i, B^j\rangle(t) - q^i(t)dB^i(t) - dN(t), \\ p(T) = & (\phi_x(\bar{x}(T)))^T. \end{cases} \tag{5.6}$$

Define the Hamiltonian $H: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1\times d} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times [0,T] \to \mathbb{R}$ as follows:

$$\begin{aligned} H(x,y,z,u,v,p,q,t) = & \ \langle p, b(t,x,u)\rangle + \langle p, h^{ij}(t,x,u)\rangle\gamma^{ij}(t) + \langle q^j, \sigma^i(t,x,u)\rangle\gamma^{ij}(t) + \langle p(f_{z_l}(t,x,y,z,v) \\ & + g_{z_l}^{ij}(t,x,y,z,v)), \sigma^l(t,x,u)\rangle\gamma^{ij}(t) + f(t,x,y,z,u) + g^{ij}(t,x,y,z,u)\gamma^{ij}(t), \end{aligned}$$

where $i, j, l = 1, \ldots, d$.

Theorem 5.2 *Suppose (H1)-(H3) hold. Let $\bar{u}(\cdot)$ be an optimal control and $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot), \bar{K}(\cdot))$ be the corresponding trajectory. Then there exist a $P^* \in \mathcal{P}^*$ and $(p(\cdot), q(\cdot), N(\cdot)) \in M_{P^*}^2(0, T; \mathbb{R}^n) \times M_{P^*}^2(0, T; \mathbb{R}^{n \times d}) \times \mathcal{M}_{P^*}^{2,\perp}(0, T; \mathbb{R}^n)$, which is the solution of the adjoint equation (5.6), such that*

$$H_u(\bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t), \bar{u}(t), p(t), q(t), t)(u - \bar{u}(t)) \geq 0, \ \forall u \in U, \ a.e., \ P^* - a.s.. \tag{5.7}$$

In the following, we give the sufficient condition for optimality.

Theorem 5.3 *Suppose (H1)-(H3) hold. Let $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ and $P^* \in \mathcal{P}^*$ satisfy that*

$$H_u(\bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t), \bar{u}(t), p(t), q(t), t)(u - \bar{u}(t)) \geq 0, \ \forall u \in U, \ a.e., \ P^* - a.s.,$$

where $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot), \bar{K}(\cdot))$ is the state processes of (3.1) and (3.2) corresponding to $\bar{u}(\cdot)$ and $(p(\cdot), q(\cdot), N(\cdot))$ is the solution of the adjoint equation (5.6) under P^*. We also assume that H is convex with respect to x, y, z, u and ϕ is convex with respect to x. Then $\bar{u}(\cdot)$ is an optimal control.

6 LQ problem

For simplicity, we suppose $d = 1$. In this case,

$$G(a) = \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-), \ a \in \mathbb{R},$$

where $\bar{\sigma}^2 = \hat{\mathbb{E}}[(B_1)^2]$, $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-(B_1)^2]$. Consider the following LQ problem. The state equation is

$$\begin{cases} dx(t) = & [A(t)x(t) + \tilde{B}(t)u(t) + b(t)]dt + [C(t)x(t) + D(t)u(t) + \sigma(t)]dB(t), \\ x(0) = & x_0, \ x_0 \in \mathbb{R}^n, \end{cases} \tag{6.1}$$

where $\mathcal{U}[0, T] := \{u(\cdot) \mid u(\cdot) \in M_G^2(0, T; \mathbb{R}^m)\}$ and $A(\cdot), C(\cdot), \tilde{B}(\cdot), D(\cdot), b(\cdot), \sigma(\cdot)$ are deterministic functions. The cost functional is

$$J(u(\cdot)) = \frac{1}{2}\hat{\mathbb{E}}\Big[\int_0^T [\langle Q(t)x(t), x(t)\rangle + 2\langle S(t)x(t), u(t)\rangle + \langle R(t)u(t), u(t)\rangle]dt + \langle Lx(T), x(T)\rangle\Big],$$

where $Q(\cdot), S(\cdot), R(\cdot)$ are deterministic functions. The stochastic optimal control problem is to minimize the cost functional over $\mathcal{U}[0, T]$.

In the following, the variable t will be suppressed. We suppose the functions satisfy the following conditions:

$$\begin{cases} A, C \in L^\infty(0, T; \mathbb{R}^{n \times n}), \ \tilde{B} \in L^\infty(0, T; \mathbb{R}^{n \times m}), \ D \in C(0, T; \mathbb{R}^{n \times m}), \\ Q \in L^\infty(0, T; \mathbb{S}_n), \ S \in L^\infty(0, T; \mathbb{R}^{m \times n}), \ R \in C(0, T; \mathbb{S}_m), \\ b, \sigma \in L^2(0, T; \mathbb{R}^n), \ L \in \mathbb{S}_n \end{cases} \tag{6.2}$$

$$R \gg 0, \ Q - SR^{-1}S^T \geq 0, \ L \gg 0, \tag{6.3}$$

where $R \gg 0$ means that there exists a $\delta > 0$ such that $R \geq \delta I$ and similarly for $L \gg 0$. In this case, the Hamiltonian function is

$$H(x, u, p, q, t) = \langle p, Ax + \tilde{B}u + b \rangle + \langle q, Cx + Du + \sigma \rangle \gamma(t) + \frac{1}{2}(\langle Qx, x \rangle + 2\langle Sx, u \rangle + \langle Ru, u \rangle). \qquad (6.4)$$

Let $\bar{u}$ be an optimal control. By maximum principle which still holds for this case, there exists a $P^* \in \mathcal{P}$ such that

$$\begin{cases} E_{P^*}[\bar{K}(T)] = 0; \\ \\ \tilde{B}^T p(t) + D^T q(t)\gamma(t) + S\bar{x}(t) + R\bar{u}(t) = 0, \text{ under } P^*, \end{cases} \qquad (6.5)$$

where $(p(\cdot), q(\cdot), N(\cdot))$ is the solution of the following adjoint equation under the probability P^*

$$\begin{cases} -dp(t) = [Q\bar{x}(t) + S^T\bar{u}(t) + A^T p(t) + C^T q(t)\gamma(t)]dt - q(t)dB(t) - dN(t), \\ \\ p(T) = L\bar{x}(T). \end{cases} \qquad (6.6)$$

Suppose that

$$p(t) = P(t)\bar{x}(t) + \varphi(t) \qquad (6.7)$$

with $P(\cdot) \in C^1([0, T], \mathbb{S}_n)$, $\varphi(\cdot) \in C^1([0, T], \mathbb{R}^n)$. Applying Itô's formula to $p(t)$, we can get

$$q(t) = P(t)C(t)\bar{x}(t) + P(t)D(t)\bar{u}(t) + P(t)\sigma(t),$$

$$\dot{P}\bar{x} + PA\bar{x} + P\tilde{B}\bar{u} + Pb + \dot{\varphi} + Q\bar{x} + S^T\bar{u} + A^T p + C^T q\gamma = 0.$$

Combining (6.5), (6.7) and the above two equalities, we can obtain that

$$q = [PC - PD(R + D^T PD\gamma)^{-1}(\tilde{B}^T P + S + D^T PC\gamma)]\bar{x} - PD(R + D^T PD\gamma)^{-1}(\tilde{B}^T \varphi + D^T P\sigma\gamma) + P\sigma, \quad (6.8)$$

$$\bar{u} = -(R + D^T PD\gamma)^{-1}[(\tilde{B}^T P + S + D^T PC\gamma)\bar{x} + \tilde{B}^T \varphi + D^T P\sigma\gamma], \qquad (6.9)$$

and the following Riccati equation for P

$$\begin{cases} \dot{P} + PA + A^T P + C^T PC\gamma + Q \\ -(\tilde{B}^T P + S + D^T PC\gamma)^T(R + D^T PD\gamma)^{-1}(\tilde{B}^T P + S + D^T PC\gamma) = 0, \text{ a.e. } t \in [0, T], \qquad (6.10) \\ P(T) = L, \end{cases}$$

$$\begin{cases} \dot{\varphi} + [A - \tilde{B}(R + D^T PD\gamma)^{-1}(\tilde{B}^T P + S + D^T PC\gamma)]^T \varphi \\ +[C - D(R + D^T PD\gamma)^{-1}(\tilde{B}^T P + S + D^T PC\gamma)]^T P\sigma\gamma + Pb = 0, \text{ a.e. } t \in [0, T], \qquad (6.11) \\ \varphi(T) = 0. \end{cases}$$

It is important to note that P^* is uniquely determined by the choose of γ. We choose $\gamma(t) = \bar{\sigma}^2$. It is well known that the Riccati equation (6.10) has a unique solution $P \gg 0$, and then equation (6.11) has a unique solution φ. In this case, the optimal control

$$\bar{u} = -(R + D^T PD\bar{\sigma}^2)^{-1}[(\tilde{B}^T P + S + D^T PC\bar{\sigma}^2)\bar{x} + \tilde{B}^T \varphi + D^T P\sigma\bar{\sigma}^2], \qquad (6.12)$$

23

where

$$\begin{cases} d\bar{x}(t) = & [A(t)\bar{x}(t) + \tilde{B}(t)\bar{u}(t) + b(t)]dt + [C(t)\bar{x}(t) + D(t)\bar{u}(t) + \sigma(t)]dB(t), \\ \\ x(0) = & x_0, \ x_0 \in \mathbb{R}^n. \end{cases} \tag{6.13}$$

In the following, we prove that the above $\bar{u}$ is the optimal control.

Theorem 6.1 *Suppose (6.2) and (6.3) hold. Then $\bar{u}$ defined in (6.12) and (6.13) is the optimal control, where P and φ are solutions for equations (6.10) and (6.11) with $\gamma(t) = \bar{\sigma}^2$.*

Proof. Let $P^* \in \mathcal{P}$ be the probability such that $\langle B \rangle(t) = \bar{\sigma}^2 t$. It is easy to check that p, q defined in (6.7) and (6.8), and $N = 0$ is the solution of the adjoint equation (6.6) under the probability P^*. Also, it is easy to check that the Hamiltonian function H is convex with respect to x, u and

$$H_u(\bar{x}(t), \bar{u}(t), p(t), q(t), t) = 0, \ \text{under } P^*.$$

By Theorem 4.9, we only need to verify that $E_{P^*}[\bar{K}(T)] = 0$. Let l be the solution of the following ODE:

$$\begin{cases} l' + \langle \varphi, b \rangle + \frac{1}{2}\bar{\sigma}^2 \langle P\sigma, \sigma \rangle \\ \\ -\frac{1}{2}(\tilde{B}^T\varphi + \bar{\sigma}^2 D^T P\sigma)^T (R + D^T PD\bar{\sigma}^2)^{-1}(\tilde{B}^T\varphi + \bar{\sigma}^2 D^T P\sigma) = 0, \ \text{a.e. } t \in [0, T], \\ \\ l(T) = 0. \end{cases}$$

Set

$$\tilde{Y}(t) = \frac{1}{2}\langle P\bar{x}, \bar{x} \rangle + \langle \varphi, \bar{x} \rangle + l,$$

$$\tilde{Z}(t) = \langle P\bar{x} + \varphi, C\bar{x} + D\bar{u} + \sigma \rangle,$$

$$\tilde{K}(t) = \frac{1}{2}\int_0^t \langle P(C\bar{x} + D\bar{u} + \sigma), C\bar{x} + D\bar{u} + \sigma \rangle d\langle B \rangle(s) - \int_0^t G(\langle P(C\bar{x} + D\bar{u} + \sigma), C\bar{x} + D\bar{u} + \sigma \rangle)ds.$$

By applying Itô's formula to $\tilde{Y}$ and some simple calculations, we can get

$$\tilde{Y}(t) = \frac{1}{2}\langle L\bar{x}(T), \bar{x}(T) \rangle + \frac{1}{2}\int_t^T [\langle Q\bar{x}, \bar{x} \rangle + 2\langle S\bar{x}, \bar{u} \rangle + \langle R\bar{u}, \bar{u} \rangle]ds - \int_t^T \tilde{Z}(s)dB(s) - (\tilde{K}(T) - \tilde{K}(t)),$$

which implies that $\bar{K}(T) = \tilde{K}(T)$. Note that $\langle P(C\bar{x} + D\bar{u} + \sigma), C\bar{x} + D\bar{u} + \sigma \rangle \geq 0$, then we get

$$\bar{K}(T) = \frac{1}{2}\int_0^T \langle P(C\bar{x} + D\bar{u} + \sigma), C\bar{x} + D\bar{u} + \sigma \rangle d(\langle B \rangle(s) - \bar{\sigma}^2 s).$$

Obviously, $E_{P^*}[\bar{K}(T)] = 0$. Thus $\bar{u}$ is the optimal control. ∎

Remark 6.2 *Using the same method, we can obtain the result for the state equation and cost functional containing the term $\langle B \rangle$. For the Riccati equation (6.10), we only need $R + D^T PD\bar{\sigma}^2 > 0$. This case will be discussed in our forthcoming paper. Note that $R + D^T PD\underline{\sigma}^2 < 0$ may be hold, so the LQ problem may be infinite for some $P \in \mathcal{P}$, but it is finite under G-expectation. The reason of this is the uncertainty of probability measures, which is different from classical LQ problem.*

24

In the following, we give an example to point out that the LQ problem with random coefficients is more difficult and P^* is not the probability measure such that $\langle B \rangle(t) = \bar{\sigma}^2 t$.

Example 6.3 *We consider the following 1-dimensional state equation:*

$$x(t) = \int_0^t \sqrt{as - \langle B \rangle(s)} dB(s),$$

where $a > \bar{\sigma}^2$ is a constant. The cost functional is

$$J(u(\cdot)) = \frac{1}{2}\hat{\mathbb{E}}\Big[\int_0^T (at - \langle B \rangle(t))|u(t)|^2 d\langle B \rangle(t) + |x(T)|^2\Big].$$

By applying Itô's formula to $|x(t)|^2$, it is easy to check that

$$J(u(\cdot)) = \frac{1}{2}\hat{\mathbb{E}}\Big[\int_0^T (at - \langle B \rangle(t))(|u(t)|^2 + 1)d\langle B \rangle(t)\Big].$$

Obvious, the optimal control $\bar{u} \equiv 0$ and $P^* \in \mathcal{P}$ satisfies

$$\hat{\mathbb{E}}\Big[\int_0^T (at - \langle B \rangle(t))d\langle B \rangle(t)\Big] = E_{P^*}\Big[\int_0^T (at - \langle B \rangle(t))d\langle B \rangle(t)\Big].$$

By simple calculation, we can obtain P^* is the probability measure such that

$$\langle B \rangle(t) = \int_0^t (\underline{\sigma}^2 I_{[0,t^*]}(s) + \bar{\sigma}^2 I_{(t^*,T]}(s))ds,$$

where $t^* = \bar{\sigma}^2 T(a + \bar{\sigma}^2 - \underline{\sigma}^2)^{-1}$. It is easy to check that this $\bar{u}$ satisfies the maximum principle in Theorem 4.8.

7 Appendix

The following proposition is about some further estimates for Theorems 4.3 and 4.6, which is interest of itself.

Proposition 7.1 *Suppose (H1)-(H3) hold. Then*

(1) *for each $u \in \mathcal{U}[0,T]$, there exists a $P^u \in \mathcal{P}^*$ such that*

$$E_{P^u}[\Theta^u] = \sup_{P \in \mathcal{P}^*} E_P[\Theta^u],$$

$$\lim_{\rho \to 0} E_{P^u}\Big[\frac{-\int_0^T m(t)dK_\rho^u(t)}{\rho}\Big] = 0;$$

(2) *there exists a $P^* \in \mathcal{P}^*$ such that*

$$\sup_{P \in \mathcal{P}^*} \inf_{u \in \mathcal{U}[0,T]} E_P[\Theta^u] = \inf_{u \in \mathcal{U}[0,T]} E_{P^*}[\Theta^u],$$

$$\inf_{u \in \mathcal{U}[0,T]} \Big(\lim_{\rho \to 0} E_{P^*}\Big[\frac{-\int_0^T m(t)dK_\rho^u(t)}{\rho}\Big]\Big) = 0.$$

Proof. (1) Consider

$$y_\rho^u(t) - \bar{y}(t) = \quad \phi(x_\rho^u(T)) - \phi(\bar{x}(T)) + \int_t^T (f_\rho^u(s) - f(s))ds - \int_t^T (z_\rho^u(s) - \bar{z}(s))dB(s)$$

$$-(K_\rho^u(T) - K_\rho^u(t)) + (\bar{K}(T) - \bar{K}(t)).$$

By Theorem 4.3, there exists a $P^u \in \mathcal{P}^*$ such that $E_{P^u}[\Theta^u] = \sup_{P \in \mathcal{P}^*} E_P[\Theta^u]$. Note that $\bar{K} \equiv 0$ under probability P^u. Similar as in the proof of Theorem 4.3, we can derive

$$E_{P^u}\left[\frac{\int_0^T m(s)dK_\rho^u(s)}{\rho}\right] = E_{P^u}[\Theta^u] - \frac{y_\rho^u(0) - \bar{y}(0)}{\rho} + o(1).$$

By Theorem 4.3,

$$\lim_{\rho \to 0} \frac{y_\rho^u(0) - \bar{y}(0)}{\rho} = E_{P^u}[\Theta^u],$$

which implies that

$$\lim_{\rho \to 0} E_{P^u}\left[\frac{-\int_0^T m(s)dK_\rho^u(s)}{\rho}\right] = 0.$$

(2) For any $P \in \mathcal{P}^*$, similar analysis as in (1), we have

$$E_P\left[\frac{\int_0^T m(s)dK_\rho^u(s)}{\rho}\right] = E_P[\Theta^u] - \frac{y_\rho^u(0) - \bar{y}(0)}{\rho} + o(1).$$

Then,

$$\lim_{\rho \to 0} E_P\left[\frac{-\int_0^T m(s)dK_\rho^u(s)}{\rho}\right] = E_{P^u}[\Theta^u] - E_P[\Theta^u] \geq 0. \tag{7.1}$$

By Minimax Theorem, we can get

$$\inf_{u \in \mathcal{U}[0,T]} \sup_{P \in \mathcal{P}^*} E_P[\Theta^u] = \sup_{P \in \mathcal{P}^*} \inf_{u \in \mathcal{U}[0,T]} E_P[\Theta^u].$$

By the proof of Theorem 4.6, we can obtain a $P^* \in \mathcal{P}^*$ such that

$$\sup_{P \in \mathcal{P}^*} \inf_{u \in \mathcal{U}[0,T]} E_P[\Theta^u] = \inf_{u \in \mathcal{U}[0,T]} E_{P^*}[\Theta^u].$$

Note that

$$\inf_{u \in \mathcal{U}[0,T]} \sup_{P \in \mathcal{P}^*} E_P[\Theta^u] = \inf_{u \in \mathcal{U}[0,T]} E_{P^u}[\Theta^u],$$

$$E_{P^u}[\Theta^u] = \sup_{P \in \mathcal{P}^*} E_P[\Theta^u] \geq E_{P^*}[\Theta^u].$$

We deduce that

$$\inf_{u \in \mathcal{U}[0,T]} (E_{P^u}[\Theta^u] - E_{P^*}[\Theta^u]) = 0.$$

Taking $P = P^*$ in (7.1), it yields that

$$\inf_{u \in \mathcal{U}[0,T]} \left(\lim_{\rho \to 0} E_{P^*}\left[\frac{-\int_0^T m(s)dK_\rho^u(s)}{\rho}\right]\right) = 0.$$

This completes the proof. ∎

References

[1] M. Avellaneda, A. Levy and A. Paras, *Pricing and hedging derivative securities in markets with uncertain volatilities*, Appl. Math. Finance, 2(1995), pp. 73–88.

[2] R. Buckdahn, B. Labedb, C. Rainera and L. Tamer, *Existence of an optimal control for stochastic control systems with nonlinear cost functional, Stochastics*, An International Journal of Probability and Stochastics Processes, 82(2010), pp. 241–256.

[3] Z. Chen and L. Epstein, *Ambiguity, risk, and asset returns in continuous time*, Econometrica, 70(2002), pp. 1403-1443.

[4] L. Denis, M. Hu and S. Peng, *Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths,* Potential Anal., 34(2011), pp. 139-161.

[5] D. Duffie and L. Epstein, *Stochastic differential utility*, Econometrica, 60(1992), pp. 353–394.

[6] D. Duffie and C. Skiadas, *Continuous-Time Security Pricing: A Utility Gradient Approach*, Journal of Mathematical Economics, 23(1994), pp. 107-131.

[7] El Karoui and S. Huang, *A general result of existence and uniqueness of backward stochastic differential equations*, El Karoui, N. and Mazliak, L. eds., Paris, 1995–1996, Pitman Research Notes in Mathematics Series, 364, 27-36, Longman, Harlow, 1997.

[8] N. El Karoui, S. Peng and M. C. Quenez, *Backward stochastic differential equations in finance*, Math. Finance, 7(1997), pp. 1-71.

[9] N. El Karoui, S. Peng and M. C. Quenez, *A dynamic maximum principle for the optimization of recursive utilities under constraints*, Ann. Appl. Probab. 11(2001), pp. 664–693.

[10] L. Epstein and S. Ji, *Ambiguous Volatility, Possibility and Utility in Continuous Time,* (2013), To appear in Journal of Mathematical Economics.

[11] L. Epstein and S. Ji, *Ambiguous volatility and asset pricing in continuous time*, Rev. Finan. Stud., 26 (2013), pp. 1740-1786.

[12] M. Fuhrman, Y. Hu and G. Tessitore, *Stochastic maximum principle for optimal control of SPDEs*, Appl. Math. Optim. 68 (2013), pp. 181–217.

[13] M. Hu, *Direct method on stochastic maximum principle for optimization with recursive utilities,* (2015), arXiv:1507.03567.

[14] M. Hu and S. Ji, *Dynamic Programming Principle for Stochastic Recursive Optimal Control Problem under G-framework,* (2014), arXiv:1410.3538.

[15] M. Hu, S. Ji, S. Peng and Y. Song, *Backward Stochastic Differential Equations Driven by G-Brownian Motion*, Stochastic Processes and their Applications, 124(2014), pp. 759-784.

[16] M. Hu, S. Ji, S. Peng and Y. Song, *Comparison Theorem, Feynman-Kac Formula and Girsanov Transformation for BSDEs Driven by G-Brownian Motion*, Stochastic Processes and their Applications, 124(2014), pp. 1170-1195.

[17] M. Hu and S. Peng, *On representation theorem of G-expectations and paths of G-Brownian motion*, Acta Math. Appl. Sin. Engl. Ser., 25(2009), pp. 539-546.

[18] Y. Hu and S. Peng, *Maximum principle for optimal control of stochastic system of functional type*, Stochastic Anal. Appl. 14 (1996), pp. 283–301.

[19] Q. Lu and X. Zhang, *General Pontryagin-type stochastic maximum principle and backward stochastic evolution equations in infinite dimensions*, Springer Briefs in Mathematics. Springer, Cham, 2014.

[20] T. Lyons, *Uncertain volatility and the risk-free synthesis of derivatives,* Appl. Math. Finance, 2(1995), pp. 117–133.

[21] T. Meyer-Brandis, B. Oksendal and X. Y. Zhou, *A mean-field stochastic maximum principle via Malliavin calculus*, Stochastics 84 (2012), pp. 643–666.

[22] E. Pardoux and S. Peng, *Adapted Solutions of Backward Stochastic Equations,* Systerm and Control Letters, 14(1990), pp. 55-61.

[23] S. Peng, *A general stochastic maximum principle for optimal control problems*, SIAM J. Control Optim., 28 (1990), pp. 966-979.

[24] S. Peng, *Backward stochastic differential equations and applications to optimal control*, Applied Mathematics and Optimization, 27(1993), pp. 125-144.

[25] S. Peng, *G-expectation, G-Brownian Motion and Related Stochastic Calculus of Itô type*, Stochastic analysis and applications, Abel Symp., 2, Springer, Berlin, (2007), pp. 541-567.

[26] S. Peng, *G-Brownian Motion and Dynamic Risk Measure under Volatility Uncertainty*, (2007), arXiv:0711.2834v1.

[27] S. Peng, *Multi-Dimensional G-Brownian Motion and Related Stochastic Calculus under G-Expectation*, Stochastic Processes and their Applications, 118(2008), pp. 2223-2253.

[28] S. Peng, *Nonlinear Expectations and Stochastic Calculus under Uncertainty*, (2010), arXiv:1002.4546v1.

[29] J. Qiu and S. Tang, *Maximum principle for quasi-linear backward stochastic partial differential equations*, J. Funct. Anal. 262 (2012), pp. 2436–2480.

[30] H. M. Soner, N. Touzi and J. Zhang, *Wellposedness of Second Order Backward SDEs,* Probability Theory and Related Fields, 153(2012), pp. 149-190.

[31] S. Tang, *The maximum principle for partially observed optimal control of stochastic differential equations*, SIAM J. Control Optim. 36 (1998), pp. 1596–1617.

[32] Z. Wu, *A general maximum principle for optimal control of forward–backward stochastic systems*, Automatica, 49(2013), pp. 1473-1480.

[33] Y. Xu, *Stochastic maximum principle for optimal control with multiple priors*, System and Control Letter, 64(2014), pp. 114-118.

[34] J. Yong, *Optimality variational principle for controlled forward–backward stochastic differential equations with mixed initial-terminal conditions*, SIAM Journal on Control and Optimization, 48(2010), pp. 4119–4156.

[35] J. Yong and X. Y. Zhou, *Stochastic controls: Hamiltonian systems and HJB equations*, 1999.

[36] X. Y. Zhou, *The connection between the maximum principle and dynamic programming in stochastic control*, Stochastics Stochastics Rep. 31 (1990), pp. 1–13.